Managing Inequality over Business Cycles: Optimal Policies with Heterogeneous Agents and Aggregate Shocks*

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Abstract

We present a truncation theory of idiosyncratic histories for heterogeneous-agent models. This method allows us to solve for optimal Ramsey policies in such models with aggregate shocks. The method can be applied to a large variety of settings, with occasionally-binding credit constraints. We use this theory to characterize the optimal level of unemployment insurance over the business cycle in a production economy.

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1 Introduction

Incomplete insurance market economies provide a useful framework for examining many relevant aspects of inequalities and individual risks. In these models, infinitely-lived agents face incomplete insurance markets and borrowing limits that prevent them from perfectly hedging their idiosyncratic risk, in line with the Bewley-Huggett-Aiyagari literature (Bewley, 1983; Imrohoroğlu, 1989; Huggett, 1993; Aiyagari, 1994; Krusell and Smith, 1998). These frameworks are now widely used, since they fill a gap between micro- and macroeconomics, and enable the inclusion of aggregate shocks and a number of additional frictions on both the goods and labor markets. However, little is known about optimal policies in these environments due to the difficulties generated by the large and time-varying heterogeneity across agents. This is unfortunate, since a vast literature suggests that the interaction between wealth heterogeneity and capital accumulation has first-order implications for the design of optimal policies. An important example is the optimal design of time-varying unemployment benefits in an economy with fluctuating unemployment risk, which has not yet been studied in the general case due to the difficulties generated by the variations in precautionary savings over the business cycle.

We present a general method that enables us to compute the steady-state allocations of the Ramsey equilibria, as well as to simulate the dynamics with aggregate shocks using perturbation method. In standard incomplete insurance market economies, agents differ according to the full history of their idiosyncratic risk realizations. Huggett (1993) and Aiyagari (1994), using the results of Hopenhayn and Prescott (1992), have shown that economies without aggregate risk have a recursive structure when the distribution of wealth is introduced as a state variable. Unfortunately, the distribution of wealth has an infinite number of possible values, which is at the root of many difficulties.

The main idea of our resolution method is to go back to the sequential formulation of incomplete-market models to construct a consistent finite state-space representation. We proceed in three steps. First, we construct a partition in the space of idiosyncratic histories, using a truncation procedure. For a given truncation length N, all agents with the same idiosyncratic history over the last N periods are grouped together. Second, we provide an exact aggregation theory of the model with aggregate shocks along these truncated histories.[1] As a result, the allocations of the full-fledged model are expressed in terms of aggregate variables for truncated histories (i.e., groups of agents sharing the same N-period history), rather than individual ones. This results in a so-called *exact aggregated model*, which is an equivalent representation of the full-fledged model and does not involve any simplifying assumptions. Finally, from this aggregated model, we construct a truncated model, based on one additional simplifying assumption: the within-history heterogeneity is assumed to remain constant in the dynamics. The truncated model thus focuses on the dynamics of heterogeneity across truncated histories.

[1] More general partitions – based on other criteria than the truncation – could be considered in a very similar way. It could rely on any criterion guaranteeing a partition of the space of idiosyncratic histories.

The interest of this construction is twofold. First, we can prove that the Ramsey allocation of the truncated model converges to the one of the full-fledged incomplete-market model with aggregate shocks when the truncation length becomes infinitely long.[2] To do so, we use the tools developed in dynamic contracts, sometimes referred to as the Lagrangian approach and developed by Marcet and Marimon (2019). We show how the Lagrangian approach must be adapted to deal with our model and occasionally-binding credit constraints, where some regularity conditions may not hold. This convergence result proves that the truncation methodology provides a consistent representation. Second, our methodology provides a simple numerical procedure to solve for Ramsey allocations both at the steady state and with aggregate shocks. In particular, the finite state-space structure allows us to use linear algebra to compute steady-values of Lagrange multipliers in closed-form. The dynamics can be simulated with perturbation methods and rely on standard softwares, such as Dynare. We find that a short truncation length yields an accurate solution.

We use our truncation theory to characterize optimal unemployment benefits over the business cycle in an economy where agents face both productivity risk and time-varying employment risk, as in Krueger et al. (2018). The economy is hit by aggregate shocks that affect technology and labor market transitions. Agents choose their labor supply when working, consume, save, and face incomplete markets for idiosyncratic risk and credit constraints. In this economy, a planner chooses the level of unemployment benefits in each period, which must be fully financed by a distorting labor tax. Although the economic trade-off is the standard trade-off between insurance and efficiency, this problem is very hard to solve in a general equilibrium setting. The level of unemployment benefits directly affects agents' welfare as well as their saving decisions and the dynamics of interest rates and wages. We find that the replacement rate is countercyclical, increasing the transfer to unemployed agents in recessions. This policy reduces the volatility of the total income of unemployed agents, what is welfare improving.

Literature review. Our paper contributes to the recent literature on optimal policies in heterogeneous agent models. Aiyagari (1995) presents an initial paper studying the Ramsey allocation in a general setup, with a characterization of the optimal capital tax. Other papers, such as Aiyagari and McGrattan (1998) or Krueger and Ludwig (2016), derive optimal policies by maximizing the aggregate steady-state welfare rather than by determining the optimal Ramsey policy, which does not account for the welfare cost of transitions. Açikgöz (2015), further developed in Açikgöz et al. (2018), uses an explicit Lagrangian approach to derive the planner's first-order conditions at the steady state and relies on a numerical procedure to approximate the value of Lagrange multipliers. Dyrda and Pedroni (2018) and Chang et al. (2018) compute optimal policies without considering

[2] We also check that, in terms of numerical performance, the truncation method delivers dynamics similar to those implied by other numerical techniques, in particular those of Reiter (2009) and Boppart et al. (2018).

the planner's first-order conditions, and instead directly maximize the intertemporal welfare over all possible paths for the planner's instruments. This method is computationally very intensive, which limits the number of possible applications. Nuño and Moll (2018) consider a continuous-time framework in which they use the techniques of Ahn et al. (2017) to simplify the derivation of the planner's first-order conditions. All these papers solve for optimal policy without aggregate shocks. Our truncation method allows us to solve for optimal policies with no restrictions, both at the steady state and with aggregate shocks. To the best of our knowledge, the only paper deriving optimal Ramsey policy in a general environment with incomplete insurance markets and aggregate shocks is Bhandari et al. (2020). Their method can account for large aggregate shocks but relies on a "primal approach" in which credit constraints can be always binding or never binding, but not occasionally binding. Compared to their model, our solution strategy works well with occasionally binding credit constraints, which may be the relevant case in some environments.[3]

Our paper also contributes to the literature on solution methods for incomplete insurance market economies with aggregate shocks. Our truncation method is related to other projection and perturbation methods (Rios-Rull, 1999, Reiter, 2009, and Young, 2010), which have been shown to be accurate approximations compared to global solution techniques (see Boppart et al., 2018 or Auclert et al., 2019). The main difference is that our solution is based on idiosyncratic histories and not on the space of wealth, which turns out to be helpful for solving Ramsey programs. In particular, our method keeps track of the relevant distribution, which is necessary for deriving optimal policies, which depend on the distributive effects of the Ramsey instruments. Finally, our truncation of idiosyncratic histories is related to, but different from, the truncation of aggregate histories (see for instance Chien et al., 2011, 2012). Our truncation is used to derive a limited-heterogeneity representation of the full-fledged model, which we simulate using perturbation methods. To the best of our knowledge, this paper is the first to develop a truncation in the space of idiosyncratic histories.

Finally, regarding the application, our paper contributes to the literature on optimal unemployment benefits. This literature is huge and a large part of it employs the sufficient-statistics approach (see the surveys of Chetty, 2009, Chetty and Finkelstein, 2013, and Kolsrud et al., 2018 for recent developments), based on partial-equilibrium analysis. A handful of papers introduce general equilibrium effects, such as Mitman and Rabinovich (2015), Landais et al. (2018a,b), or Ábrahám et al. (2019), but they focus on labor market externalities and not on saving distortions. To the best of our knowledge, the only paper analyzing optimal unemployment insurance in general equilibrium with saving choices is Krusell et al. (2010). To simplify the quantitative exercise, the authors perform a welfare

[3]Another strategy used in the literature is to focus on a simplified economy, in which the wealth distribution has only one or two mass points. This solution generates a tractable equilibrium (see McKay and Reis, 2020, Bilbiie and Ragot, 2020, Ravn and Sterk, 2017, and Challe, 2020, among others). These models provide important economic insights but they cannot identify certain properties related to the time-varying wealth distribution. Their quantitative relevance is thus hard to assess.

analysis by comparing steady states with different levels of unemployment benefits. We adopt a different approach, deriving the time-varying solution of a general Ramsey problem in an economy with aggregate shocks.

The rest of the paper is organized as follows. In Section 2 we present the environment.In Section 3 we derive optimal Ramsey policies and discuss the economic trade-off for optimal unemployment benefits over the business cycle. In Section 4 we construct the truncated model and provide convergence properties. Section 5 sets out our quantitative analysis.

2 The economy

Time is discrete and indexed by $t = 0, 1, 2, \ldots$ The economy is populated by a continuum of agents of measure 1, distributed on an interval $\mathcal{I}$ according to a measure $\ell(\cdot)$. We follow Green (1994) and assume that the law of large numbers holds.

2.1 Preferences

In each period, agents derive utility from private consumption c and disutility from labor l. The period utility function, denoted by $U(c,l)$, is assumed to be of the Greenwood-Hercowitz-Huffman (GHH) type, exhibiting no wealth effect for the labor supply, as in Heathcote (2005), for instance:

$$U(c,l) = u\left(c - \chi^{-1}\frac{l^{1+1/\varphi}}{1+1/\varphi}\right), \tag{1}$$

where $\varphi > 0$ is the Frisch elasticity of labor supply, $\chi > 0$ scales labor disutility, and $u : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously derivable, increasing, and concave, with $u'(0) = \infty$. Our results do not rely on the GHH functional form and we could consider a more general utility function U. The algebra is simplified, however, especially in the Ramsey program, because of the absence of a wealth effect for the labor supply. In Appendix E.3, we show how to use our truncation method with a more general utility function.

Agents have standard additive intertemporal preferences, with a constant discount factor $0 < \beta < 1$. They therefore rank consumption and labor streams, denoted respectively by $(c_t)_{t\geq 0}$ and $(l_t)_{t\geq 0}$, using the intertemporal utility criterion $\sum_{t=0}^{\infty}\beta^t U(c_t, l_t)$.

2.2 Risks

We consider a general setup where agents face an aggregate risk, a time-varying unemployment risk, and a productivity risk, as modeled by Krueger et al. (2018). As will be clear in the quantitative analysis below, this general setup allows us to match realistic labor market wealth distribution and dynamics.[4]

[4]Compared to Krueger et al. (2018), we introduce endogenous labor supply, such that labor taxes are distorting. In addition, we simplify the economy and remove the age dimension. We follow these authors and denote all transitions by Π – which will be distinguished by their subscripts.

Aggregate risk. The aggregate risk affects both aggregate productivity and unemployment risk. At a given date t, the aggregate state is denoted by z_t and takes values in the (possibly continuous) state space $\mathcal{Z} \subset \mathbb{R}^+$. We assume that the aggregate risk is a Markov process. The history of aggregate shocks up to time t is denoted by $z^t = \{z_0, \ldots, z_t\} \in \mathcal{Z}^{t+1}$. For the sake of clarity, we will denote the realization of any random variable $X_t : \mathcal{Z}^{t+1} \to \mathbb{R}$ in state z^t by X_t, instead of $X_t(z^t)$, when there is no ambiguity.

Employment risk. At the beginning of each period, each agent $i \in \mathcal{I}$ faces an uninsurable idiosyncratic employment risk, denoted by e_t^i at date t. The employment status e_t^i can take two values, e and u, corresponding to employment and unemployment, respectively. We denote the set of possible employment statuses by $\mathcal{E} = \{e, u\}$. An employed agent with $e_t^i = e$ can freely choose her labor supply l_t^i. An unemployed agent with $e_t^i = u$ cannot work and will receive an unemployment benefit financed by a distorting tax on labor and will suffer from a fixed disutility reflecting a domestic effort. These aspects are further described below.

The employment status $(e_t^i)_{t\geq 0}$ follows a discrete Markov process with transition matrix $\Pi(z^t) \in [0,1]^{2\times 2}$ – that will simply denoted as Π_t –, which is assumed to depend on the history of aggregate shocks up to date t. The job-separation rate between periods $t-1$ and t is denoted by $\Pi_{t,eu} = 1 - \Pi_{t,ee}$, while $\Pi_{t,ue} = 1 - \Pi_{t,uu}$ is the job-finding rate between $t-1$ and t. We denote the implied population shares of unemployed and employed agents by $S_{t,u}$ and $S_{t,e}$, respectively, with $S_{t,u} + S_{t,e} = 1$.

Productivity risk. Agents' individual productivity, denoted by y_t^i, is stochastic and takes values in a finite set $\mathcal{Y} \subset \mathbb{R}_+$. Large values in $\mathcal{Y}$ correspond to high productivities. The before-tax wage earned by an employed agent i is the product of the aggregate wage w_t (dependent on aggregate shock), the labor effort l_t^i, and individual productivity y_t^i. The total before-tax wage is therefore $y_t^i w_t l_t^i$. An unemployed agent will also carry an idiosyncratic productivity level that will affect her unemployment benefits and her disutility level, denoted by ζ_y (for productivity $y \in \mathcal{Y}$), associated with domestic production.

The productivity status follows a first-order Markov process where the transition probability from state $y_{t-1}^i = y$ to $y_t^i = y'$ is constant and denoted by $\Pi_{yy'}$. In particular, it is independent of the agent's employment status. We denote by S_y the share of agents endowed with individual productivity level y. This share is constant over time because of the assumptions regarding transition probabilities $(\Pi_{yy'})$.

The individual state of any agent i is characterized by her employment status and her productivity level. We denote by $s_t^i = (e_t^i, y_t^i)$ the date-t individual status of any agent, whose possible values lie in the set $\mathcal{S} = \mathcal{E} \times \mathcal{Y}$. Finally, we denote by $s^{i,t} = \{s_0^i, \ldots, s_t^i\}$ a history until period t. We can then use the transition probabilities for employment and productivity to derive the measure $\mu_t : \mathcal{S}^{t+1} \to [0,1]$ – $\mu_t(s^t)$ being the measure of agents with history s^t in period t.

2.3 Production

The good is produced by one profit-maximizing representative firm. This firm is endowed with production technology that transforms, at date t, labor L_t (in efficient units) and capital K_{t-1} into Y_t output units of the single good. The production function F is a Cobb-Douglas function with parameter $\alpha \in (0,1)$ featuring constant returns-to-scale. Capital must be installed one period before production and the total productivity factor Z_t is stochastic. Constant capital depreciation is denoted by $\delta > 0$, and net output Y_t is formally defined as follows:

$$Y_t = F(Z_t, K_{t-1}, L_t) = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1}, \tag{2}$$

where the total productivity factor is the exponential of the aggregate shock z_t: $Z_t = \exp(z_t)$.

The two factor prices at date t are the aggregate before-tax wage rate w_t and the capital return r_t. The profit maximization of the producing firm implies the following factor prices:

$$w_t = F_L(Z_t, K_{t-1}, L_t) \text{ and } r_t = F_K(Z_t, K_{t-1}, L_t). \tag{3}$$

2.4 Unemployment insurance

A benevolent government manages an unemployment insurance (UI) scheme, in which labor taxes are raised to finance unemployment benefits. As labor supply is endogenous, labor tax is distorting. The government thus faces the standard trade-off between efficiency and insurance.

At any date t, unemployed agents receive an unemployment benefit that is equal to a constant fraction of the wage the agent would earn if she were employed (with the same productivity level). The replacement rate is denoted by ϕ_t and the unemployment benefit of an agent i endowed with productivity y_t equals $\phi_t w_t y_t \bar{l}_t\,(y_t)$, where $\bar{l}_t\,(y_t)$ is the average labor supply of employed agents with productivity y_t. We follow Krueger et al. (2018) for this specification, which usefully reduces the state space. From the agents' perspective, the replacement rate is an exogenous process that depends on the aggregate state $\phi_t = \phi_t(z^t)$.

Unemployment benefits are financed solely by the labor tax, which is only paid by employed agents. Taxes amount to a constant share τ_t of employed agents' wages with this proportion being identical for all employed agents. The contribution τ_t is set such that the UI scheme budget is balanced at any date t, no social debt being allowed:

$$\phi_t w_t \int_{i \in \mathcal{U}_t} y_t^i \bar{l}_t^i(y_t^i) \ell(di) = \tau_t w_t \int_{i \in \mathcal{I} \setminus \mathcal{U}_t} y_t^i l_t^i \ell(di), \tag{4}$$

where $\mathcal{U}_t \subset \mathcal{I}$ is the set of unemployed agents at t and $\mathcal{I} \setminus \mathcal{U}_t$ is the set of employed agents.

2.5 Agents' program and resource constraints

We consider an agent $i \in \mathcal{I}$. She can save in an asset that pays the gross interest rate $1 + r_t$. She is prevented from borrowing too much and her savings must remain above an exogenous threshold, $-\bar{a} \leq 0$. At date 0, the agent chooses the consumption $(c_t^i)_{t\geq 0}$, labor supply $(l_t^i)_{t\geq 0}$, and saving plans $(a_t^i)_{t\geq 0}$ that maximize her intertemporal utility, subject to a budget constraint and the previous borrowing limit. Formally, for a given initial wealth a_{-1}^i, her program is:[5]

$$\max_{\{c_t^i, l_t^i, a_t^i\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\Big(c_t^i - \chi^{-1} \frac{l_t^{i,1+1/\varphi} 1_{e_t^i=e} + \zeta_{y_t^i}^{1+1/\varphi} 1_{e_t^i=u}}{1+1/\varphi}\Big), \tag{5}$$

$$c_t^i + a_t^i = (1+r_t)a_{t-1}^i + ((1-\tau_t)l_t^i 1_{e_t^i=e} + \phi_t \bar{l}_t(y_t^i) 1_{e_t^i=u}) y_t^i w_t, \tag{6}$$

$$a_t^i \geq -\bar{a},\ c_t^i > 0,\ l_t^i > 0. \tag{7}$$

Objective (5) accounts for the disutility of unemployed agents associated with domestic production. The budget constraint (6) is standard and the expression $((1-\tau_t)l_t^i 1_{e_t^i=e} + \phi_t \bar{l}_t(y_t^i) 1_{e_t^i=u}) y_t^i w_t$ is a compact formulation for the net wage (i.e., after taxes and unemployment benefits).

We denote by $\beta^t \nu_t^i$ the Lagrange multiplier on the credit constraint of agent i. The Lagrange multiplier is obviously null when the agent is not credit constrained. Taking advantage of the GHH utility function, the first-order conditions of an employed agent's program (5)–(7) are:

$$u'(c_t^i - \chi^{-1} \frac{\hat{l}_t^{i,1+1/\varphi}}{1+1/\varphi}) = \beta \mathbb{E}_t \Big[(1+r_{t+1}) u'(c_{t+1}^i - \chi^{-1} \frac{\hat{l}_{t+1}^{i,1+1/\varphi}}{1+1/\varphi})\Big] + \nu_t^i, \tag{8}$$

$$(l_t^i)^{1/\varphi} = \chi(1-\tau_t) w_t y_t^i, \tag{9}$$

where for all $t \geq 0$ and $i \in \mathcal{I}$, we introduce the notation $\hat{l}_t^i$:

$$\hat{l}_t^i \equiv l_t^i 1_{e_t^i=e} + \zeta_{y_t^i} 1_{e_t^i=u}. \tag{10}$$

The GHH utility function implies that the labor supply in equation (9) only depends on current productivity and the after-tax real wage, which implies: $\bar{l}_t(y_t^i) = l_t^i$. Unemployed agents have the same Euler equation (8). They supply no labor, but they earn unemployment benefits and suffer from disutility (terms in ζ_y) related to home production.

We now turn to economy-wide constraints. Financial- and labor-market clearing implies the following relationships for the supply of capital K_t and labor L_t (in efficient units):

$$\int_i a_t^i \ell(di) = K_t \text{ and } L_t = \int_{i \in \mathcal{I} \setminus \mathcal{U}_t} y_t^i l_t^i \ell(di). \tag{11}$$

[5] In the remainder of the paper, 1_A will denote an indicator function equal to 1 if A is true and 0 otherwise. For any $t \geq 0$, $\mathbb{E}_t$ will denote an expectation operator, conditional on the information available at date t.

The clearing of the goods market implies that total consumption and the new capital stock equals total supply, itself the sum of output net of depreciation and past capital:

$$\int_i c_t^i \ell(di) + K_t = Y_t + K_{t-1}. \tag{12}$$

Using labor market transition probabilities, we deduce that the law of motion for the employed and unemployed agent populations, denoted respectively by $S_{t,e}$ and $S_{t,u}$, is:

$$S_{t,u} = 1 - S_{t,e} = \Pi_{t,eu} S_{t-1,e} + \Pi_{t,uu} S_{t-1,u}. \tag{13}$$

The constant share of agents S_y with productivity y verifies: $S_y = \sum_{y \in \mathcal{Y}} S_{y'} \Pi_{y'y}$.

Using individual labor Euler conditions (9), the UI budget constraint (4) can be written as: $\phi_t \int_{i \in \mathcal{U}_t} \left(y_t^i\right)^{1+\varphi} \ell(di) = \tau_t \int_{i \in \mathcal{I} \setminus \mathcal{U}_t} \left(y_t^i\right)^{1+\varphi} \ell(di)$. We observe that the budget balance only depends on the current idiosyncratic state. With equation (13), this can therefore be simplified into $\phi_t \sum_{y \in \mathcal{Y}} S_{t,u} S_y y^{1+\varphi} = \tau_t \sum_{y \in \mathcal{Y}} S_{t,e} S_y y^{1+\varphi}$, or:

$$\phi_t S_{t,u} = \tau_t S_{t,e}. \tag{14}$$

We can now formulate our equilibrium definition.

Definition 1 (Sequential equilibrium) *A competitive equilibrium is a collection of individual variables $(c_t^i, l_t^i, a_t^i, \nu_t^i)_{t \geq 0, i \in \mathcal{I}}$, of aggregate quantities $(K_t, L_t, Y_t)_{t \geq 0}$, of price processes $(w_t, r_t)_{t \geq 0}$, and of UI policy $(\tau_t, \phi_t)_{t \geq 0}$ such that, for an initial wealth distribution $\left(a_{-1}^i\right)_{i \in \mathcal{I}}$, and for initial values of capital stock $K_{-1} = \int_i a_{-1}^i \ell(di)$, and of the aggregate shock z_{-1}, we have:*

1. *given prices, individual strategies $\left(c_t^i, l_t^i, a_t^i, \nu_t^i\right)_{t \geq 0, i \in \mathcal{I}}$ solve the agent's optimization program in equations (5)–(7);*

2. *financial, labor, and goods markets clear: for any $t \geq 0$, equations (11) and (12) hold;*

3. *the UI budget is balanced: equation (14) holds for all $t \geq 0$;*

4. *factor prices $(w_t, r_t)_{t \geq 0}$ are consistent with the firm's program (3).*

3 Ramsey program

3.1 Formulation of the Ramsey program

The Ramsey problem involves determining the unemployment insurance policy (which consists here of the replacement rate ϕ_t and the labor tax rate τ_t) that corresponds to the "best" competitive equilibrium, according to a utilitarian welfare criterion. Aggregate welfare is simply measured by the sum $\sum_{t=0}^{\infty} \beta^t \int_i U(c_t^i, l_t^i) \ell(di)$. The Ramsey problem can thus be written as – with $\hat{l}_t^i$ defined in (10):

$$\max_{((a_t^i,c_t^i,l_t^i)_{i\in\mathcal{I}},\phi_t,\tau_t,r_t,w_t)_{t\geq 0}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\int_i U(c_t^i,\hat{l}_t^i)\ell(di)\right], \tag{15}$$

subject to: (i) individual budget constraints (6), (ii) the Euler equations (8) and (9) for consumption and labor, respectively, (iii) the UI scheme budget balance (14), (v) the market clearing constraints (11), and finally (vi) the factor prices definitions (3).

A reformulation of the Ramsey problem. We simplify the formulation of the Ramsey problem exposed in equation (15), using the factorization of the Lagrangian employed by Marcet and Marimon (2019). However, two difficulties arise when applying this approach in our environment. The first difficulty is the consistency of the method with heterogeneous-agent models. In fact, we show in Appendix B.1 that our setup adds no complexity to the formulation of the planner's objective. The second difficulty is the application of Marcet and Marimon (2019) to models with occasionally binding credit constraints. To show that the first-order conditions of the Lagrangian approach are valid, we derive an additional theoretical result in Proposition 6 of Appendix E.2, where we prove that the first-order conditions of our Ramsey problem can be understood as the limit of those of a Ramsey problem featuring penalty functions (whose concavity become infinitely high). Since penalty functions substitute for credit constraints, this result shows that Marcet and Marimon (2019)'s result still applies with occasionally binding credit constraints.[6] We here directly provide the first-order conditions of our Ramsey problem to simplify the exposition.

Notation. We denote by $\beta^t\lambda_t^i$ the Lagrange multiplier of the consumption Euler equation (8) for agent i at date t. These Lagrange multipliers are key to understanding the planner's program. If agent's i private incentives to save at date t are socially optimal, then her Euler equation is not a constraint and the Lagrange multiplier is $\lambda_t^i = 0$. Depending on how the planner perceives the saving distortions, these coefficients can be either positive or negative. A positive (resp. negative) Lagrange multiplier for agent i reflects that the planner perceives that agent i saves too little (resp. too much). We provide an example in Appendix E.1 to clarify this aspect. Finally, the multiplier λ_t^i is null when the credit constraint is binding for agent i. The product $\lambda_t^i\nu_t^i$ (for any t and i) is thus always null.

3.2 Ramsey conditions and economic interpretation

Using proper substitution, the Ramsey program (15) can be written as a maximization problem with only two sets of choice variables: the labor tax τ_t and saving choices $(a_t^i)_{t,i}$. See Appendix B.1 for a simplified expression of the Ramsey problem. The current section

[6] We thank Albert Marcet for this suggestion.

derives the planner's first-order conditions and discusses the economic trade-offs that determine the time-varying replacement rate.

To ease the economic interpretation of the first-order conditions, we define:[7]

$$\psi_t^i = U_c(c_t^i, \hat{l}_t^i) - (\lambda_t^i - (1 + r_t)\lambda_{t-1}^i)U_{cc}(c_t^i, \hat{l}_t^i), \tag{16}$$

which will be called the *marginal social valuation of liquidity* for agent i, because it is the marginal gain for the planner of transferring resources to agent i at date t. If agent i receives one additional unit of goods today, this additional unit will have a private value proportional to U_c. The planner also has to account for the effect on the saving incentives, i.e., on the Euler equations. This additional unit therefore affects the agent's saving incentive from period $t-1$ to period t and from period t to period $t+1$. This effect is captured by the second term, which is proportional to U_{cc}.

The saving decision a_t^i in history s^N at date t affects the individual welfare of all agents due to general equilibrium effects on capital and prices. The first-order condition of the Ramsey program related to saving choices summarizes all of these effects. It can be written as follows for an unconstrained agent i:[8]

$$\begin{aligned}\psi_t^i = &\underbrace{\beta\mathbb{E}_t\left[(1 + r_{t+1})\psi_{t+1}^i\right]}_{\text{liquidity smoothing}} + \underbrace{\beta\frac{\alpha K_t^{-1}}{1+\alpha\varphi}\mathbb{E}_t\left[\int_i \psi_{t+1}^i \overbrace{(1-\tau_{t+1})y_{t+1}^i w_{t+1} l_{t+1}^i 1_{e_{t+1}^i = e}}^{=\text{net wage}}\ell(di)\right]}_{\text{wage effect for employed}} \\ &+ \underbrace{\beta\frac{\alpha K_t^{-1}}{1+\alpha\varphi}\mathbb{E}_t\left[\int_i \psi_{t+1}^i (1+\varphi)\overbrace{\phi_{t+1} y_{t+1}^i w_{t+1} l_{t+1,e}^i 1_{e_{t+1}^i = u}}^{=\text{unemp. benefits}}\ell(di)\right]}_{\text{wage effect (unemp. ben.) for unemployed}} \\ &- \underbrace{\beta\frac{\alpha K_t^{-1}}{1+\alpha\varphi}\mathbb{E}_t\left[\frac{w_{t+1}L_{t+1}}{K_t}\int_i \left(\lambda_t^i U_c(c_{t+1}^i, \hat{l}_{t+1}^i) + \psi_{t+1}^i a_t^i\right)\ell(di)\right]}_{\text{interest rate effect on smoothing and wealth}}\Bigg\}.\end{aligned} \tag{17}$$

Equation (17) features the first-order condition on the liquidity allocation (i.e., saving choices) for unconstrained agents. Although it appears complicated, the equation has a straightforward interpretation. Four effects are at play. The first is a direct effect that measures the expected future value of liquidity tomorrow. In other words, this component states that liquidity value should be smoothed over time. This first part is very similar to a standard consumption Euler equation, except that it reflects "social" marginal utilities ψ_t^i, rather than "private" marginal utilities U_c^i. We refer to this first term as "liquidity smoothing". The three other components alter the pure smoothing effect and reflect the fact that the planner also takes into account the consequences of liquidity allocation on

[7] Unlike Chien et al. (2011) or Marcet and Marimon (2019), we do not use cumulative Lagrange multipliers to analyze the dynamics. Instead, we use the period multipliers to derive the planner's first-order conditions. These conditions are easier to interpret and the simulation of the model relies on a smaller number of variables. See these two references for a discussion of the existence of these multipliers in such economies.

[8] Proofs for first-order conditions can be found in Appendix B.1.

prices. More precisely, the second and third components correspond to the marginal effect of additional saving on the wage rate. This affects employed agents (second component) and unemployed (third component) agents, because UI benefits are proportional to the labor income of employed agents with the same productivity. Finally, the fourth and last component reflects the distortions through the interest rate on saving incentives.

The second first-order condition, relating to the labor tax, can be written as follows:

$$\underbrace{\frac{S_{t,e}}{S_{t,u}}\left(\frac{1}{\varphi}+1-\frac{1-\alpha}{1-\tau_t}\right)\int_i \psi_t^i \frac{l_t^i y_t^i}{L_t} 1_{e_t^i=u}\ell(di)}_{\text{gain of unemployment benefits for unemployed}} = \tag{18}$$

$$\underbrace{\frac{1}{\varphi}\int_i \psi_t^i \frac{l_t^i y_t^i}{L_t} 1_{e_t^i=e}\ell(di)}_{\text{cost of the tax for employed}} + \underbrace{\frac{\alpha}{(1-\tau_t)K_{t-1}}\int_i \left(\lambda_{t-1}^i U_c(c_{t+1}^i, \hat{l}_{t+1}^i) + \psi_{t+1}^i a_{t-1}^i\right)}_{\text{effect on prices, smoothing, and redistribution}}.$$

Equation (18) determines the optimal labor tax rate by setting the marginal costs of a higher tax rate equal to the marginal benefits. On the left-hand side of equation (18), the marginal benefit comprises the marginal gain of tax (and UI benefit) for unemployed agents. On the right-hand side of (18), marginal costs comprise two effects. The first effect accounts for the impact of the labor tax on employed agents, taking into account the negative net effect on the labor supply (inversely proportional to the Frisch elasticity φ). The second one reflects the tax distortion on the interest rate and thus on saving incentives. Note that equation (18) embeds, in a compact form, the general equilibrium effect on wages, which are captured by both the Frisch elasticity of the labor supply, φ, and the concavity of the production function, α.

4 The aggregation and truncation theory

This section presents our methodology in four steps. First, Section 4.1 explains how the full-fledged model can be expressed using groups of agents rather than individual agents. Agents are grouped together if they share the same idiosyncratic history over the last N periods (where $N > 0$ is exogenously set). In other words, individual allocations are aggregated along the N-period history of agents. This procedure is exact and does imply any simplifying assumptions. It results in a so-called *exact aggregated model*.

Second, Section 4.2 describes how to simplify the aggregated model to approximate the dynamics of the incomplete-market model. The simplification involves an assumption that states that heterogeneity within truncated history is constant and does not vary with aggregate shocks. The resulting model is called the *truncated model*, which can be simulated using perturbation methods.

The two last steps of this section discusses the benefits of our methodology. Section 4.5 states that the Ramsey allocations in the truncated model converge to those of the full-fledged incomplete-market model with aggregate shocks. Section 4.6 describes the algorithm we use to simulate the Ramsey allocation with aggregate shocks.

4.1 Aggregation of an incomplete-market model

The equilibrium of the full-fledged model with aggregate shocks of Section 2 can be characterized by a set of policy rules (for saving, consumption and labor supply) and Lagrange multipliers (for the borrowing limit), defined over all idiosyncratic and aggregate histories. Formally:

$$a_t(s^t, z^t),\ c_t(s^t, z^t),\ l_t(s^t, z^t),\ \nu_t(s^t, z^t),\ t \geq 0,\ s^t \in \mathcal{S}^t,\ z^t \in \mathcal{Z}^t$$

are solutions of the equilibrium characterized by equations (6)–(9). The main idea of the aggregation procedure is to group together agents with the same idiosyncratic history over the last N periods – where $N > 0$ is the *truncation length.* Such an history – that will be called a *truncated history* – is a vector $s^N \in \mathcal{S}^N$. Considering an agent i with (full) idiosyncratic history $s^{i,t}$ at date $t \geq N$, her truncated history can be represented as:

$$s^{i,t} = \{\ \ldots, s^t_{-N-2}, s^t_{-N-1}, s^t_{-N}, \underbrace{s^t_{-N+1}, \ldots, s^t_{-1}, s^t_0}_{=s^N}\}, \tag{19}$$

where s^t_{-k} is her idiosyncratic status (at date t) k periods in the past. We now explain how to construct the aggregated model.

First, we need to compute the measure of agents with truncated history s^N. An agent having a truncated history $\hat{s}^N$ at $t-1$ will have a different truncated history s^N at t depending on the realization of the idiosyncratic risk at date t. The probability of transitioning from $\hat{s}^N$ at $t-1$ to s^N at t is denoted by $\Pi_{t,\hat{s}^N s^N}$ (with $\sum_{s^N \in \mathcal{S}^N} \Pi_{t,\hat{s}^N s^N} = 1$) and can be inferred from the transition probabilities for unemployment and productivity:

$$\Pi_{t,\hat{s}^N s^N} = 1_{s^N \succeq \hat{s}^N} \Pi_{t,\hat{e}^N_0, e^N_0} \Pi_{\hat{y}^N_0 y^N_0} \geq 0, \tag{20}$$

where $s^N_0 = (e^N_0, y^N_0)$ and $\hat{s}^N_0 = (\hat{e}^N_0, \hat{y}^N_0)$ are the current idiosyncratic states for s^N and $\hat{s}^N$, respectively. We can deduce from these transition probabilities the share of agents in the population endowed with history s^N at date t, that is denoted by S_{t,s^N}:

$$S_{t,s^N} = \sum_{\hat{s}^N \in \mathcal{S}^N} S_{t-1,\hat{s}^N} \Pi_{t,\hat{s}^N s^N}, \tag{21}$$

where the initial shares $(S_{-1,s^N})_{s^N \in \mathcal{S}^N}$, with $\sum_{s^N \in \mathcal{S}^N} S_{-1,s^N} = 1$, are given.

Second, the aggregated model involves computing consumption, saving and labor choices for groups of agents with the same truncated history s^N. Consider a generic variable, that we denote by $X_t(s^t, z^t)$, and that possibly depends on the histories s^t of idiosyncratic risk and z^t of aggregate risk. For a truncated history s^N, the aggregation of X at date t, denoted by X_{t,s^N} is defined as the following average:

$$X_{t,s^N} = \frac{1}{S_{t,s^N}} \sum_{s^t \in \mathcal{S}^t \big| (s^t_{t-N+1}, \ldots, s^t_t) = s^N} X_t(s^t, z^t) \mu_t(s^t), \tag{22}$$

where we recall that $\mu_t(s^t)$ is the measure of agents with history s^t. With the definition (22), c_{t,s^N}, a_{t,s^N}, l_{t,s^N}, and ν_{t,s^N} are respectively the average consumption, the end-of-period saving, the labor supply and the credit-constraint Lagrange multiplier among agents having at date t the truncated history s^N.

Third, we have to compute aggregate beginning-of-period wealth. Because agents switch histories from one period to another, the beginning-of-period wealth for history s^N at date t is derived from period-$(t-1)$ end-of-period wealth and from transition across histories. Indeed, it consists of the wealth of all agents having history s^N in period t and any other possible history in $t-1$. Formally, the beginning-of-period wealth $\tilde{a}_{t,s^N}$ for truncated history s^N is:

$$\tilde{a}_{t,s^N} = \sum_{\hat{s}^N \in \mathcal{S}^N} \frac{S_{t-1,\hat{s}^N}}{S_{t,s^N}} \Pi_{t,\hat{s}^N,s^N} a_{t-1,\hat{s}^N}. \tag{23}$$

Fourth, the method involves the aggregation of Euler equations. As the marginal utility is not linear in consumption in the general case, the marginal utility of consumption aggregation is different from the aggregation of marginal utility. If we denote the latter by u'_{t,s^N} for truncated history s^N at date t, we formally have: $u'(c_{t,s^N}) \neq u'_{t,s^N}$. As these two quantities are scalars, we can compute their ratio that will be denoted by ξ_{t,s^N}. The parameters $(\xi_{t,s^N})_{t,s^N}$ guarantee that Euler equations hold with aggregate consumption levels.[9] As a consequence, the ξs can be seen as the relevant summary of within-truncated-history heterogeneity for the dynamics of the aggregated model. Indeed, we show in the numerical investigation of Section 5 that the ξs efficiently capture this within-history heterogeneity in a parsimonious way, even with a short truncation, as $N = 2$.[10]

Using the previous steps of the aggregation mechanism, we are able to characterize the dynamics of the exact aggregated model. We obtain the following set of equations:

$$c_{t,s^N} + a_{t,s^N} \leq (1+r_t)\tilde{a}_{t,s^N} + \left((1-\tau_t)1_{e_0^N=e} l_{t,s^N} + \phi_t 1_{e_0^N=u} l_{t,s^N,e}\right) y_0^N w_t, \tag{24}$$

$$\xi_{t,s^N} U_c(c_{t,s^N}, \hat{l}_{t,s^N}) = \beta \mathbb{E}_t (1+r_{t+1}) \sum_{\tilde{s}^N \in \mathcal{S}^N} \Pi_{t+1,s^N,\tilde{s}^N} \xi_{t+1,\tilde{s}^N} U_c(c_{t+1,\tilde{s}^N}, \hat{l}_{t+1,\tilde{s}^N}) + \nu_{t,s^N}, \tag{25}$$

$$l_{t,s^N}^{\frac{1}{\varphi}} = \chi(1-\tau_t) w_t y_0^N, \tag{26}$$

$$\hat{l}^i_{t,s^N} = l_{t,s^N} 1_{e_t^i=e} + \zeta_{y_t^i} 1_{e_t^i=u}. \tag{27}$$

The budget constraint (24) is the aggregation of individual budget constraints (6) using

[9] This argument explains why, thanks to the GHH utility function, there is no such truncation wedge for the labor supply Euler equation because the latter is linear in current productivity (see equation (9)). However, it would be present with a more general utility function and correcting coefficients would be needed for both Euler equations. Though slightly more involved, it is noteworthy that this would not impair the construction of the truncated economy. See Appendix E.3 for further details.

[10] These preference parameters are related to the aggregation procedure presented in Werning (2015). The difference is that Werning (2015) captures the heterogeneity through a change in the discount factor, whereas we capture with preference parameters affecting utility in each period. This is only important to derive Ramsey allocations.

equations (22) and (23). The same holds for the Euler labor equation (26) that comes from individual Euler equation (9). Finally, the aggregated Euler equation (24) for consumption relies on the parameters (ξ_{t,s^N}) that enable the aggregation of individual Euler equations (8). The system (24)–(27) is an exact aggregation of the full-fledged model with aggregate shocks in terms of truncated idiosyncratic histories. It characterizes the dynamics of the aggregated variables c_{t,s^N}, a_{t,s^N}, l_{t,s^N} and ν_{t,s^N} without involving any approximation.

Finally, market clearing conditions can also be expressed in terms of aggregated variables. For capital and labor we have:

$$K_t = \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} a_{t,s^N}, \quad L_t = \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} y_{s^N} l_{t,s^N}. \tag{28}$$

This aggregate market clearing conditions are exactly identical to those expressed with individual variables, in equation (11).

Aggregation at the steady-state. From the previous aggregation theory, we can state the following Proposition.

Proposition 1 (Constructing the ξs) *The preference parameters $(\xi_{s^N})_{s^N \in \mathcal{S}^N}$ can be computed at the steady state.*

The proof is provided in Appendix C.2.1 and the logic is as follows. At the steady state, we can characterize the stationary wealth distribution of the full-fledged model. From this, we can identify the set of credit-constrained histories. We can also use equation (22) to compute aggregate variables involved in the Euler equations (25) (but the ξs). These equations (25) can then be inverted to compute the preference parameters $(\xi_{s^N})_{s^N \in \mathcal{S}^N}$. We provide in equation (58) of Appendix C.2.1 a closed-form expression for the ξs.

4.2 The truncated model

Using the aggregate model, we can construct the so-called *truncated model*, which is an aggregate model in which we plug two assumptions to allow for simulations with aggregate shocks. These assumptions can be stated as follows.

Assumption A *We make the following two assumptions.*

1. *The* preference parameters $(\xi_{s^N})_{s^N}$ *remain constant and equal to their steady-state values.*
2. *The set of credit-constrained histories, denoted by $\mathcal{C} \subset \mathcal{S}^N$, is time-invariant.*

Assumption A describes how the aggregation can be simplified using these steady-state ξs to simulate the model in the presence of aggregate shocks. In other words, Assumption A allows us to rely on the aggregation procedure twice: (i) first, exactly, to compute the

ξs from steady-state allocations; (ii) second, approximately, to compute allocations with aggregate shocks using the steady-state ξs.

More precisely, the first item of Assumption A means that the model features within-history heterogeneity, but that this heterogeneity is not time-varying, as the preference parameters $(\xi_{s^N})_{s^N}$, determined at the steady state, remain constant in the presence of aggregate shocks. We thus assume the model dynamics is computed while considering that within-truncated-history heterogeneity is not-time-varying. This assumption is in the same vein as the assumption that the within-bin heterogeneity is uniform in the histogram approach of Reiter (2009).

The second item of Assumption A states that if a history $s^N \in \mathcal{S}^N$ is credit constrained at the steady state, it also remains credit constrained in the dynamic version of the model. This restriction is imposed by the perturbation method and is not specific to our construction.[11] Despite this assumption, the number of credit-constrained households can be time-varying, since the size of agents having any truncated history is time-varying.

Assumption A enable us to formally characterize the truncated model. It is defined by the set of equations (24)–(27), together with: (i) the additional assumptions that $\xi_{t,s^N} = \xi_{s^N}$ for all t; and (ii) the fact that Euler equations (24) only hold for the non-credit-constrained histories determined at the steady state. The definition can be stated as follows.

Definition 2 (Truncated equilibrium) *A truncated equilibrium is a collection of individual variables* $(c_{t,s^N}, l_{t,s^N}, a_{t,s^N}, \nu_{t,s^N})_{t\geq N-1}^{s^N\in\mathcal{S}^N}$*, of aggregate quantities* $(K_t, L_t, Y_t)_{t\geq N-1}$*, of price processes* $(w_t, r_t)_{t\geq N-1}$*, and of UI policy* $(\tau_t, \phi_t)_{t\geq N-1}$*, such that, for an initial wealth distribution* $(a_{N-2,s^N})_{s^N\in\mathcal{S}^N}$*, and for initial values of capital stock* $K_{N-2} = \sum_{s^N\in\mathcal{S}^N} a_{N-2,s^N}$*, and of the aggregate shock* z_{N-2} *we have:*

1. $(\xi_{s^N})_{s^N\in\mathcal{S}^N}$ *are the steady-state preference parameters of the aggregated model;*
2. *individual strategies* $(c_{t,s^N}, l_{t,s^N}, a_{t,s^N}, \nu_{t,s^N})_{t\geq N-1, s^N\in\mathcal{S}^N}$ *solve the agent's optimization program in equations (24)–(10).;*
3. *financial, labor, and goods markets clear: for any* $t \geq N-1$*, equations (28) hold;*
4. *the UI budget is balanced: equation (4) holds for all* $t \geq N-1$*;*
5. *factor prices* $(w_t, r_t)_{t\geq N-1}$ *are consistent with the firm's program (3).*

By construction of the parameters $(\xi_{s^N})_{s^N}$, the allocations of the aggregated and of the truncated equilibria coincide with each other at the steady state. This implies that aggregate quantities and prices at the steady state are the same in the truncated economy and the underlying Bewley (full-fledged) economy). This is stated in Corollary 1.

[11] In particular, it could be relaxed by using history-specific penalty functions to model (possibly large) aggregate shocks. We leave this development for future work.

Corollary 1 (Price and aggregate quantities) *Consider a truncated economy, where the coefficients* $(\xi_{s^N})_{s^N \in \mathcal{S}^N}$ *have been constructed following Proposition 1, based on the underlying Bewley equilibrium. Prices and aggregate quantities (aggregate consumption, total labor supply, and capital) in the truncated economy are then identical to those of the Bewley economy.*

Another property of the truncated equilibrium that we discuss further in Section 4.5 is that the allocations in the truncated equilibrium (with aggregate shocks) can be made arbitrarily close to those of the full-fledged equilibrium, when the length N of the history truncation becomes increasingly long.

4.3 Decentralization of the truncated model

We have derived the truncated model based on the aggregation of a full-fledged incomplete-market model. It is noteworthy that the truncated model can be derived from an alternative micro-foundation, which also brings additional results and insights. The truncated model can be shown to be constructed from an explicit partial insurance mechanism, where individual agents can only insure against idiosyncratic shocks occurring $N+1$ periods ago. As a consequence, agents only differ from each other according to realizations of the idiosyncratic shock over the last N periods. In this alternative construction, the parameters (ξ_{s^N}) are preference shocks that are history-specific. To save some space, this alternative construction is presented in Appendix A using an island metaphor (see Lucas, 1975, 1990, or Heathcote et al., 2017 for a more recent reference). One of the interests of this alternative construction is to prove that the dynamic of the truncated model is well-defined and can be written in a recursive form.

4.4 The Ramsey problem

We now use the truncated model to compute optimal Ramsey policies in a heterogeneous-agent model with aggregate shocks. There are two main gains of computing Ramsey allocations in the truncated model rather than aggregating first-order conditions of the full-fledged Ramsey program. First, we can prove convergence results of Ramsey allocations in the truncated model toward the true allocation, when the truncation length becomes increasingly long. Second, this strategy ensures the numerical stability of dynamic simulations, as we solve for Ramsey policies in a well-defined model (see Appendix A).

Following Section 3, the truncated Ramsey problem can be written as:

$$\max_{((a_{t,s^N}, c_{t,s^N}, l_{t,s^N}, \hat{l}_{t,s^N})_{s \in \mathcal{S}^N}, \phi_t, \tau_t, r_t, w_t)_{t \geq 0}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \xi_{s^N} U(c_{t,s^N}, \hat{l}_{t,s^N}) \right], \tag{29}$$

subject to: (i) the budget constraints (24), (ii) the Euler equations (25)–(26), the market clearing constraints (28), and (iv) the factor prices (3) and the UI budget balance (14).

As in the full-fledged program, we define the social value of liquidity Ψ_{t,s^N} as:

$$\Psi_{t,s^N} = \xi_{s^N} U_c(c_{t,s^N}, \hat{l}_{t,s^N}) - (\lambda_{t,s^N} - (1+r_t)\tilde{\lambda}_{t,s^N})\xi_{s^N} U_{cc}(c_{t,s^N}, \hat{l}_{t,s^N}), \tag{30}$$

where $\tilde{\lambda}_{t,s^N}$ is the aggregation of previous-period Lagrange multipliers $\lambda_{t-1,\tilde{s}^N}$ and is defined similarly to $\tilde{a}_{t,s^N}$ in equation (23):

$$\tilde{\lambda}_{t,s^N} = \sum_{\tilde{s}^N \in \mathcal{S}^N} \frac{S_{t-1,\tilde{s}^N}}{S_{t,s^N}} \Pi_{t,\tilde{s}^N s^N} \lambda_{t-1,\tilde{s}^N}. \tag{31}$$

The parameter Ψ_{t,s^N} is the exact parallel for histories of ψ_t^i defined in (16) for agents. Similarly to the full-fledged case, two first-order conditions need to be computed. With respect to savings, we obtain for unconstrained histories $s^N \in \mathcal{S}^N \setminus \mathcal{C}$:

$$\begin{aligned}\Psi_{t,s^N} = {} & \beta \sum_{\tilde{s}^N \in \mathcal{S}^N} \mathbb{E}_t \Big[(1+r_{t+1})\Pi_{t+1,s^N\tilde{s}^N} \Psi_{t+1,\tilde{s}^N}\Big] + \beta \frac{\alpha K_t^{-1}}{1+\alpha\varphi} \\ & \times \Bigg\{ \mathbb{E}_t \Bigg[\sum_{\tilde{s}^N \in \mathcal{S}^N} \Psi_{t+1,\tilde{s}^N} S_{t+1,\tilde{s}^N} w_{t+1} \Big((1-\tau_{t+1}) l_{t+1,\tilde{s}^N} 1_{\tilde{e}_0^N = e} + (1+\varphi)\phi_{t+1} l_{t+1,\tilde{s}^N,e} 1_{\tilde{e}_0^N = u}\Big) \tilde{y}_0^N \Bigg] \\ & - \sum_{\tilde{s}^N \in \mathcal{S}^N} \mathbb{E}_t \left[\frac{w_{t+1} L_{t+1}}{K_t} S_{t+1,\tilde{s}^N} \left(\tilde{\lambda}_{t+1,\tilde{s}^N} \xi_{\tilde{s}^N} U_c(c_{t+1,s^N}, \hat{l}_{t+1,s^N}) + \Psi_{t+1,\tilde{s}^N} \tilde{a}_{t+1,\tilde{s}^N} \right) \right] \Bigg\},\end{aligned} \tag{32}$$

while the first-order relative to the labor tax can be written as follows:

$$\begin{aligned} & \frac{S_{t,e}}{S_{t,u}} \left(\frac{1}{\varphi} + 1 - \frac{1-\alpha}{1-\tau_t} \right) \sum_{\tilde{s}^N \in \mathcal{S}^N} \Psi_{t,\tilde{s}^N} S_{t,\tilde{s}^N} \frac{l_{t,\tilde{s}^N,e}}{L_t} \tilde{y}_0^N 1_{\tilde{e}_0^N = u} = \\ & \frac{\alpha}{(1-\tau_t)K_{t-1}} \sum_{\tilde{s}^N \in \mathcal{S}^N} S_{t,\tilde{s}^N} \Big(\tilde{\lambda}_{t,\tilde{s}^N} \xi_{\tilde{s}^N} U_{c,t,s^N} + \Psi_{t,\tilde{s}^N} \tilde{a}_{t,\tilde{s}^N} \Big) + \frac{1}{\varphi} \sum_{\tilde{s}^N \in \mathcal{S}^N} S_{t,\tilde{s}^N} \Psi_{t,\tilde{s}^N} \frac{l_{t,\tilde{s}^N}}{L_t} \tilde{y}_0^N 1_{\tilde{e}_0^N = e}. \end{aligned} \tag{33}$$

It can be observed that first-order conditions (32) and (33) of the truncated Ramsey program are very similar to (17) and (18) for the full-fledged Ramsey program.

4.5 Convergence results

We provide a number of convergence results for the truncation method, that ultimately guarantee that the truncated Ramsey allocations in the presence of aggregate shocks converge to the true ones, when the history length becomes increasingly long.

Steady state. The first result states that the truncated allocations at the steady state and preference parameters. The proof can be found in Appendix B.3.

Proposition 2 (Convergence of allocations) *The steady-state allocation of the truncated model satisfies:*

$$(c_{t,s^N}, a_{t,s^N}, l_{t,s^N})_{s^N \in \mathcal{S}^N} \longrightarrow_N (c_t(s^\infty), a_t(s^\infty), l_t(s^\infty))_{s^\infty \in \mathcal{S}^\infty}, \text{ almost surely,}$$

where $(c_t(s^\infty), a_t(s^\infty), l_t(s^\infty))_{s^\infty \in \mathcal{S}^\infty}$ *are the steady-state allocations of the full-fledged* *model. Similarly, for the preference parameters, we have:* $\xi_{s^N} \longrightarrow_N 1$, *almost surely.*

This result is rather intuitive. As N increases, the agents' shared history becomes longer and the first period with a potentially different idiosyncratic status becomes more distant. This means that as N increases, the agents assigned to a given history become more "similar" and the within-history heterogeneity becomes smaller. Furthermore, since the within-history heterogeneity vanishes, the ξs have an increasingly smaller role to play and converge to 1 for large N.

Aggregate shocks. The following statement partially extends the convergence result of Proposition 2 to the presence of aggregate shocks.

Proposition 3 (Allocation convergence in the presence of aggregate shocks) *In the model with aggregate shocks, the allocations of the truncated model converge to the solution of a first-order perturbation of the full-fledged model.*

The intuition of the proof is rather straightforward. The perturbation method involves constructing the economy as a first-order polynomial of the model aggregate shocks. The coefficients of this polynomial are functions of the model truncated variables (i.e., they include the ξs), which converge toward their Bewley counterparts. The polynomial therefore converges toward the polynomial associated with the full-fledged model. Obviously, this method assumes that aggregate shocks are not too large, and can be properly solved via the perturbation methods. This property is shared with other simulation methods as Reiter (2009), Boppart et al. (2018) or Auclert et al. (2019).

Ramsey program. The following proposition states that the solution of the approximated Ramsey model converges to the solution of the actual Ramsey program, if this solution exists. This result is the Ramsey equilibria parallel of Proposition 2, which held for competitive equilibria.

Proposition 4 (Convergence of the approximate Ramsey program) *We assume that savings choices are bounded from above by* $a_{\max} > 0$. *The steady-state allocation of the Ramsey allocation on the truncated model converges toward the allocation of steady-state Ramsey allocation of the full-fledged model as the truncation length increases.*

More precisely, the limit, as N *grows, of the steady-state solutions of the Ramsey program on the truncated model* $K, L, r, w, \tau, \phi, (c_{s^N}, a_{s^N}, l_{s^N}, \lambda_{s^N})_{s^N}$, *converges almost surely to the to the allocations* $(c(s^\infty), a(s^\infty), l(s^\infty))_{s^\infty \in \mathcal{S}^\infty}$, *the Lagrange multipliers* $(\lambda(s^\infty))_{s^\infty \in \mathcal{S}^\infty}$, *the aggregate quantities* (K, L), *the factor prices* (r, w), *and the UI policy* (τ, ϕ), *respectively, such that:*

- *allocations* $(c(s^\infty), a(s^\infty), l(s^\infty))_{s^\infty \in \mathcal{S}^\infty}$ *are those of the full-fledged Bewley model;*
- *factor prices* (r, w) *are consistent with aggregate quantities* (K, L) *and verify (3);*

– *Lagrange multipliers and the UI policy are consistent with the FOC of an exact Ramsey program (30)-(18).*

The proof of this result can be found in Appendix B.5. We need a technical assumption imposing an upper bound on saving choices $a_{\max}$ (which can be arbitrarily large) which is required to apply Berge theorem. Proposition 4 states that in the absence of aggregate shocks, the solution of the approximate Ramsey program converges to the solution of an exact Ramsey program (if it exists) when the history length becomes increasingly long. The truncation can therefore asymptotically approximate not only competitive equilibria but also Ramsey equilibria.

As was the case for the competitive equilibrium, this convergence result can be extended to aggregate shocks in the context of a perturbation method. We state this result formally in the following corollary.

Corollary 2 (Convergence of the Ramsey program with aggregate shocks) *The convergence result of Proposition 4 can be extended to the presence of aggregate shocks when the Ramsey program is solved with the perturbation method (assuming that a steady-state Ramsey equilibrium exists).*

4.6 The algorithm to simulate the Ramsey allocation

We now provide the main steps of the algorithm used to approximate the Ramsey allocation at the steady state.

1. Choose a truncation length N.

2. Set an initial value for the UI policy.

3. Solve for the full-fledged Bewley model (i.e., without aggregate shocks). This gives individual allocations, aggregate quantities and prices.

4. Construct the aggregated model at the steady state:

   (a) Compute aggregate allocations $(c_{s^N}, l_{s^N}, a_{s^N})_{s^N}$.

   (b) Compute $(\xi_{s^N})_{s^N}$ using the closed-form expression in Appendix C.2.1.

   (c) Use equation (32) at the steady state to compute $(\Psi_{s^N})_{s^N}$ and Lagrange multipliers $(\lambda_{s^N})_{s^N}$. See Appendix C.2.2 for a closed-form expression.

5. Compute optimality condition (18). If it does not hold (up to a precision criterion), update the UI policy, and iterate starting at Step 3.

6. Finally, increase N and repeat Steps 2-5 until the optimal UI policy does not change.

This algorithm has three advantages. First, the computation of preference parameters ξs for the truncated model is consistent with the full-fledged Bewley model *for each UI* *policy under consideration* (as can be seen in Step 4, that follows Step 2 for each UI policy). In particular, the steady state corresponding to the optimal Ramsey UI policy is by construction also the steady state of the full-fledged Bewley model (for same UI policy). Second, our algorithm requires the Bewley model to exist for each UI policy (because of Step 3). The Ramsey problem must therefore select an existing competitive equilibrium and the perturbation method – used to compute the model with aggregate shocks – cannot be run around non-existing steady-state equilibria. Third, our method allows us to simply compute the steady-state values of Lagrange multipliers in closed-form using matrix calculus (Step 2c). This provides a new and very efficient algorithm with which to compute the steady-state solution of the Ramsey program.

Once the steady state of the model has been computed, the dynamics of the model – provided in Appendix C.1 – can be solved by perturbation techniques, as there are a finite number of equations. This means that the simulation can rely on existing software such as Dynare (Adjemian et al., 2011). This implies that the (truncated) wealth distribution and the distribution of Lagrange multipliers are used as state variables. As shown in the numerical analysis of Section 5, the planner's instruments depend on these two distributions.

We can also compare the accuracy of the truncated model with the outcomes of the simulations of incomplete-insurance market model, using current simulation techniques, including those of Reiter (2009) and Boppart et al. (2018). We show in our numerical example of Section 5 that the outcomes of the different methods are quantitatively very close to each other. This accuracy of the truncated method can be generated by a low value of N. This shows the crucial role of the ξs that efficiently capture the within-history heterogeneity.

5 Numerical analysis

We now turn to the quantitative analysis. We calibrate the model and perform two exercises. First, we simulate the model with aggregate shocks and a constant replacement rate to check the accuracy of the truncated model. Second, we compute the optimal dynamics of the replacement rate and discuss the mechanisms and the accuracy of the solution.

5.1 The calibration

5.1.1 Preferences

The period is a quarter. The discount factor is $\beta = 0.99$. The period utility function is $\log(c - \chi^{-1}\frac{l^{1+1/\varphi}}{1+1/\varphi})$. The Frisch elasticity of labor supply is set to $\varphi = 0.5$, which is the value recommended by Chetty et al. (2011) for the intensive margin in heterogeneous agent models. The scaling parameter is set to $\chi = 0.04$, which implies normalizing the aggregate labor supply, defined in (11), to 1/3.

Unemployed workers cannot choose their labor supply. Their utility function is $\log(c - \chi^{-1}\frac{\zeta_y^{1+1/\varphi}}{1+1/\varphi})$, also with $\chi = 0.04$ and $\varphi = 0.5$. We recall that ζ_y is the exogenous labor supply for home production for a worker with productivity y. For agents to be worse-off when unemployed than employed, ζ_y is set to the steady-state labor supply of a worker with productivity y.

5.1.2 Technology and TFP shock

The production function is Cobb-Douglas: $Y = ZK^{\alpha}L^{1-\alpha}$. The capital share is set to $\alpha = 36\%$ and the depreciation rate is $\delta = 2.5\%$, as in Krueger et al. (2018) among others. The TFP process is a standard process, with $Z_t = \exp(z_t)$ and $z_t = \rho_z z_{t-1} + \varepsilon_t^z$, where $\varepsilon_t^z \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_z^2)$. We use the standard values $\rho_z = 0.95$ and $\sigma_z = 0.31\%$ to obtain a deviation of the TFP shock z_t equal to 1% at the quarterly frequency (see e.g., Den Haan, 2010).

5.1.3 Idiosyncratic risk

To calibrate the uninsurable labor risk, we follow the strategy of Krueger et al. (2018) and introduce both employment and productivity risks. We adapt their calibration to our economy, which features a GHH utility function and endogenous labor supply.

Unemployment risk. For the unemployment risk, we follow Shimer (2003) and assume that the job-separation rate is constant over the business cycle, while the job-finding rate is time-varying and procyclical. We set $\Pi_{eu}^{SS} = 4.87\%$ for the average job-separation rate and $\Pi_{ue}^{SS} = 78.6\%$ for the average job-finding rate. The standard deviation of the job-finding rate is set to 6%, based on US estimates (see Abeille-Becker and Clerc, 2013 or Challe and Ragot, 2016). As the standard deviation of z_t is 1%, we assume that the job-finding rate is defined as $\Pi_{t,ue} = \Pi_{ue}^{SS} + \sigma_{ue} z_t,$ with $\sigma_{ue} = 6$.

Idiosyncratic productivity risk. Idiosyncratic productivity risk is a key ingredient for the model to generate a realistic earning and wealth distribution. We calibrate a productivity process $\log y_t = \rho_y \log y_{t-1} + \varepsilon_t^y$, with $\varepsilon_t^y \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_y^2)$. As we are considering a model with endogenous labor supply, there is a difference between the earning process and the productivity process. With the GHH utility function, the labor supply is $l_y = (w\,(1-\tau)\,\chi y)^{\varphi}$. The log of labor income yl_y is thus related to $(1+\varphi)\log y$ and accordingly depends on the value chosen for the Frisch elasticity. We calibrate the process y such that the persistence and variance of the labor income yl_y match the estimated values of Krueger et al. (2018). They estimate a process with a persistent and transitory shock on productivity. Following Boppart et al. (2018), we use persistent shocks and consider transitory shocks as measurement errors. Using a Frisch elasticity of 0.5, we compute a quarterly persistence of $\rho_y = 0.9923$ and a standard deviation of $\sigma_y = 6.60\%$, which generate, for the log of earnings, an annual persistence of 0.9695 and a variance of

$\frac{3.84\%}{1-0.9695^2}$.[12] The Rouwenhorst (1995) procedure is used to discretize the productivity process into 7 idiosyncratic states with a constant transition matrix. As agents can be either employed or unemployed, each agent can be in one of the $14 = 7 \times 2$ idiosyncratic states. Table 1 provides a summary of the model parameters.

Parameter	Description	Value
β	Discount factor	0.99
α	Capital share	0.36
δ	Depreciation rate	0.025
Π_{ue}^{SS}	Average job-finding rate	78.6%
$\bar{a}$	Credit limit	0
Π_{eu}^{SS}	Average job-separation rate	4.87%
S_u^{SS}	Steady-state unemployment rate	5.83%
ρ_z	Autocorrelation TFP	0.95
σ_z	Standard deviation TFP shock	0.31%
σ_{ue}	Cov. job find. rate with TFP	6
ρ_y	Autocorrelation idio. income	0.992
σ_y	Standard dev. idio. income	6.60%
χ	Scaling param. labor supply	0.04
φ	Frisch elasticity labor supply	0.5

Table 1: Parameter values in the baseline calibration. See text for descriptions and targets.

5.2 Steady-state equilibrium distribution

We simulate a Bewley model for a constant and exogenous replacement rate ϕ. The computational details are provided in Appendix D.1 and accuracy tests are discussed in Section 5.5 below.

In Table 2, we report the wealth distribution generated by the model and compare it to the empirical distribution. We compute a number of standard statistics – listed in the first column – including the quartiles, the Gini coefficient, and the 90-95 and 95-100 intercentiles.

The empirical wealth distribution, reported in the second and third columns of Table 2, is computed using two sources, the PSID for the year 2006 and the SCF for the year 2007. The fourth and fifth columns report the wealth distribution generated by our model with two different values for the exogenous replacement rate ϕ, set either to 50% (column 4) or to 42% (column 5). The former value of $\phi = 50\%$ corresponds to the standard value used in the literature for the US, as in Krueger et al. (2018) among others. The latter value of

[12] We follow the procedure of Footnote 19 in Krueger et al. (2018). The quarterly persistence ρ_y is such that $\rho_y^4 = 0.9695$ equals the annual persistence. The variance of the log of labor income yl_y (at the quarterly frequency) is the same as the variance at the annual frequency, so: $(1+\varphi)\frac{\sigma_y^2}{1-\rho_y^2} = \frac{0.0384}{1-0.9695^2}$.

	Data		Models	
Wealth statistics	PSID, 06	SCF, 07	$\phi = 50\%$	$\phi = 42\%$
Q1	-0.9	-0.2	0.2	0.3
Q2	0.8	1.2	1.4	1.8
Q3	4.4	4.6	6.2	6.4
Q4	13.0	11.9	19.5	18.7
Q5	82.7	82.5	71.6	68.7
90-95	13.7	11.1	16.9	16.9
95-100	36.5	36.4	32.9	32.8
Gini	0.77	0.78	0.70	0.69

Table 2: Wealth distribution in the data and in the model.

$\phi = 42\%$ corresponds to the optimal steady-state replacement rate that we compute below (see Section 5.4).

Overall, the distribution of wealth generated by the model is quite similar for the two replacement rate values and is close to the data. In particular, the model does a good job in matching the wealth distribution with a high Gini of 0.70. The concentration of wealth at the top of the distribution is higher in the data than in the model. It is known that additional model features must be introduced to match the high wealth inequality in the US, such as heterogeneous discount rates, as in Krusell and Smith (1998), or entrepreneurship, as in Quadrini (1999).

Steady-state effect of a change in the replacement rate. To better understand the effect of the replacement rate, Table 3 compares some steady-state statistics for two economies, featuring the replacement rate of either 42% or 50%. The first two rows

	$\phi(\%)$	K	C	c^u/c^e	$mean\,(\lambda)$
Economy 1	42	13.1428	0.8382	0.9823	81.02
Economy 2	50	13.1304	0.8380	0.9847	78.89
Variation (%)	$8^{\dagger}$	-0.1	-0.02	-0.24	-1.41

Note: See the text for definitions of the variables. † indicates an absolute variation, while the variation is by default a relative variation.

Table 3: Implications of a variation in the replacement rate

correspond to the two economies while the third row corresponds to the relative change for the variable of interest, reported in the columns. It can be observed that an increase in the replacement rate of 8% decreases the capital stock, K, and steady-state consumption, C, by a small amount. Due to the higher replacement rate, unemployed agents are better insured by the UI scheme and agents thus express a smaller demand for self-insurance.

They therefore save less, which diminishes total savings and capital. This also decreases the consumption inequality c^u/c^e between unemployed and employed workers. The higher replacement rate also means a the higher labor tax that deters labor supply and agents' labor earnings, which has a negative impact on savings and consumption. The last column of Table 3 reports the population average Lagrange multiplier on the credit constraint, $mean\,(\lambda)$, which is discussed below to clarify the role of the optimal replacement rate.

5.3 Model dynamics with a fixed replacement rate

We now compute the dynamics of the model with a fixed replacement rate, set to $\phi = 42\%$, and with aggregate shocks affecting both TFP and the job-finding rate. We set the truncation length to $N = 2$, which implies that we simulate $14^2 = 196$ histories. We use these steady-state allocations to compute the ξs, which ensures that the truncated model has the same truncated wealth distribution as the steady-state Bewley model. Importantly, this low value of N is enough to replicate the dynamics of the model, thanks to the ξs. The model captures relevant heterogeneity in productivity levels and transitions, as discussed in Section 4. The good quality of approximation for $N = 2$ is indirect evidence that the within-history heterogeneity has a second-order effect on the dynamics of the model – but not on the steady state – and that the time-varying heterogeneity across histories is sufficient to capture the dynamics.

For the sake of clarity, we summarize the equations of the model in Appendix C.1. We simulate the model over 10,000 periods. The dynamics of the truncated model are compared to an alternative solution method, developed by Rios-Rull (1999), Reiter (2009), and Young (2010) among others, which we call the Reiter method for brevity. This method uses a histogram representation and a perturbation technique to solve the model. The method is known to provide accurate results, when compared to the global method of Krusell and Smith (1998), as shown in Boppart et al. (2018) or in Auclert et al. (2019).[13]

The comparison of the two methods can be found in Table 4, in columns labeled (1) and (2). Column (3) reports the outcomes of a representative agent (RA) economy. This last model features a unique agent and one Euler equation. All parameters are otherwise the same. Economies (4)–(6) are discussed below. Table 4 reports, for each of the three economies, the mean and the normalized standard deviation (i.e., the standard deviation divided by the mean) of the main aggregate variables: output Y, aggregate consumption C, total labor L, and the replacement rate ϕ. The table also reports the autocorrelations and correlations for consumption and output.

Overall, the Reiter and truncated methods both generate very close statistics. The standard levels of GDP and of aggregate consumption are the same, up to an order of

[13] We also compare the results of the truncation approach to those of the Boppart et al. (2018) and Auclert et al. (2019) algorithms – hereafter BKM – for TFP shocks. The results are reported in Appendix D. The three methods (Reiter, BKM, and truncation) generate very similar results. Finally, the Reiter method can be used with bases other than histograms, such as in Winberry (2018) or Bayer et al. (2019).

Repl. rate		Exogenous			Optimal rule		
Methods		Trunc.	Reiter	RA	Trunc.	Reiter	RA
Simulations		(1)	(2)	(3)	(4)	(5)	(6)
Y	mean	1.17	1.17	1.08	1.17	1.17	1.08
	std/mean (%)	2.07	2.06	2.45	2.64	2.71	2.81
C	mean	0.84	0.84	0.80	0.84	0.84	0.80
	std/mean (%)	1.70	1.69	2.02	2.26	2.27	2.35
L	mean	0.30	0.30	0.30	0.30	0.30	0.30
	std/mean (%)	0.93	0.92	1.08	1.67	1.70	1.47
K	mean	13.14	13.14	11.09	13.14	13.14	11.09
	std/mean (%)	1.93	1.92	2.54	2.29	2.45	2.93
ϕ	mean(ϕ)(%)	42	42	42	42	42	42
	std(ϕ)(%)	0	0	0	23	23	26
$corr(C, C_{-1})$	(in %)	99.09	99.13	99.66	98.74	99.18	99.67
$corr(Y, Y_{-1})$		97.55	97.56	98.22	97.41	97.52	98.21
$corr(C, Y)$		96.45	96.22	94.60	97.18	95.30	95.22
$corr(Y, \Phi)$		0	0	0	−93.23	−96.65	−96.29

Table 4: Moments of the simulated models for different specifications and different resolution techniques.

magnitude of 10^{-4}.[14] This is not due to the fact that heterogeneity does not matter. Indeed, the RA-economy (3) significantly differs from the economies implied by the Reiter and truncated methods. For instance, the normalized standard deviation of GDP in the Reiter and truncation economies is 2.06% and 2.07%, respectively, whereas it is 2.45% in the RA-economy. More specific accuracy checks are provided in Section 5.5.

5.4 Optimal replacement rate

The optimal steady-state replacement rate is computed using the algorithm described in Section 4.6. This algorithm yields an optimal steady-state replacement rate of $\phi = 42\%$, which we used in the simulations of Section 5.3. This optimal replacement rate is obtained for $N = 2$. We have checked that we obtain the same optimal replacement rate, $\phi = 42\%$, for $N = 3$, with 2,744 histories. In this last case, the computations are very slow. For this reason, we focus on the case $N = 2$ which appears to be very accurate. Once the steady-state allocations and policy instruments of the Ramsey planner have been computed, we can deduce the dynamics of the model using perturbation techniques. More details are provided in Appendices C.1 and C.2.

[14] To quantitatively assess the role of the ξs, we have computed the dynamics of the truncated model, but with all ξs set to one. The model dynamics are then very different from those implied by the truncated or Reiter methods. For instance, the normalized standard deviation of GDP is equal to 1.9, which differs from the 2.06 and 2.07 computed by the Reiter and truncated methods, respectively.

First, the tradeoffs faced by the planner have already been presented in the discussion of Table 3. An increase in the replacement rate reduces inequality and capital accumulation. This can be seen in the last column of Table 3, which reports the average value of the Lagrange multiplier on the Euler equation. As discussed in Section 3.1 and developed in Appendix E.1, this average value is positive when the planner perceives that agents save too much. The mean is negative when the planner perceives that there are not enough savings. In both economies, the mean is positive ($mean\,(\lambda) > 0$), indicating that agents save too much on average. The mean decreases with the higher replacement rate, since agents save less due to a lower precautionary motive. Increasing the replacement rate is a distorting tool that disincentivizes savings as it decreases labor supply and consumption.

Second, the dynamic properties of the replacement rate, solved in the truncated economy, are reported in economy (4) of Table 4[15]. The average replacement rate is 42% and its standard deviation is 23%. The comparison between economies (2) (with a fixed replacement rate of 42%) and (4) shows the effect of a time-varying replacement rate. It can be seen that the replacement rate is countercyclical, since $corr(Y, \phi) = -0.93 < 0$. It increases in recessions and decreases in booms. As the replacement rate is countercyclical, labor supply, aggregate consumption, and output are more procyclical in economy (4) than in economy (2).[16]

Optimal replacement rate dynamics. We now investigate whether a relatively simple rule can reproduce the dynamics of the optimal replacement rate in the truncated economy. We simulate the model for 10,000 periods and regress the replacement rate ϕ_t on several moments of aggregate variables. We find that the following (and rather complex) rule has a very high $R^2 = 0.99999$:

$$\phi_t = (1-a_1^\phi-a_2^\phi)\phi^{ss}+a_1^\phi\phi_{t-1}+a_2^\phi\phi_{t-2}+a_0^\varepsilon\varepsilon_t^z+a_1^\varepsilon\varepsilon_{t-1}^z+a_2^\varepsilon\varepsilon_{t-2}^z+a^K(K_{t-1}-\bar{K})+\varepsilon_t^\phi, \quad (34)$$

with $(\phi^{ss}, a_1^\phi, a_2^\phi, a_0^\varepsilon, a_1^\varepsilon, a_2^\varepsilon, a^K) = (0.420, -24.477, 0.010, 0.605, -0.332, 0.367, -0.033)$.[17]

We can now plug this estimated rule into the model and simulate it using the Reiter method. The purpose of this experiment is to check that the moments of the economy featuring a carefully estimated rule for the replacement rate and simulated using the Reiter method are close to the moments generated by the Ramsey model simulated by the truncation method. The results are provided in Economy (5) of Table 4. The moments are

[15] The simulation of the economy with the optimal replacement rate implies following 7 variables $(c, a, \tilde{a}, \lambda, \tilde{\lambda}, l, S)$ for the $14^2 = 196$ histories. As a consequence, there are roughly 1500 equations, in our perturbation procedure. Using a package like Dynare, it takes around 1 minute to simulate the model.

[16] In the literature, results about the cyclicality of the replacement rate are mixed. Mitman and Rabinovich (2015) find a procyclical replacement rate, whereas Landais et al. (2018a,b) find a countercyclical one. These papers study economies without capital, but with a much more detailed model of the labor market. We instead investigate the implication of an endogenous capital stock, but with a simpler labor market structure.

[17] We also estimated a simpler rule. We regress the replacement rate ϕ_t on the technology shock z and on the first, second, and third-order moments of the wealth distribution, to see whether moments of the wealth distribution can be sufficient statistics. We find a low R^2 of 0.73. It appears that a rich time structure is necessary to reproduce the dynamics of the optimal replacement rate.

very close in both economies. This confirms that the assumption of constant within-history heterogeneity is quantitatively reasonable. Again, this is not because heterogeneity has no role to play. Economy (6) of Table 4 in fact corresponds to an RA-economy with the optimal rule of equation (34) for the replacement rate and it can be verified that its moments are significantly different from those of Economies (4) and (5).

Alternative rules. To check the optimality of the time-varying rule of equation (34), we also simulate the model using the Reiter method and compute the related aggregate welfare while changing the coefficients in the rule. We first simulate an economy with a procyclical replacement rate, where we change the signs of a_0^ε, a_1^ε, a_2^ε, and a^K. The results are reported in Appendix D.3.We find that aggregate welfare decreases with this new rule: a number of agents now have a lower consumption level and a higher marginal utility in recessions. Second, we simulate the economy with a modified rule featuring the same cyclicality as the original rule, but a higher variance – such that the standard deviation of ϕ_t is now 35% instead of 23%. Welfare is again decreasing because the replacement rate falls considerably in good times and the consumption of unemployed agents consequently falls. From this experience, we can be confident of the rule's optimality – and thereby of the Ramsey program implied by the truncation method. We also better understand the role of the optimal rule in the business cycle, which attempts to stabilize the consumption of low-utility agents.

5.5 Convergence and additional accuracy tests

Convergence properties. We now report some convergence statistics when the length of the truncation increases. For any given N, we compute $(\xi_{s^N})_{s^N \in \mathcal{S}^N}$ as explained in Section 4 and then deduce the standard deviation $std(\xi)$ across histories. The values of $std(\xi)$ for $N = 2$ to $N = 4$ are reported in Table 5. It can be observed that the

Truncation length N	Number of histories	$std(\xi)$ (%)
2	196	34.1
3	2,744	32.2
4	38,416	32.1

Table 5: Convergence properties of the truncated model for different truncation lengths N.

standard deviation $std(\xi)$ decreases with N, but at a slow rate.[18] As the number of idiosyncratic states is high, equal to 14, the number of histories grows rapidly with N, which computationally limits the maximal truncation length. As explained above, our choice of N is not based on the minimization of the standard deviations of ξ. Indeed, we

[18]More generally, simulating different economies, we find that these standard deviations decrease faster when the persistence of the idiosyncratic shocks is low, not necessarily when the number of idiosyncratic states is small.

have shown in Section 5.3 that the case $N = 2$ accurately captures the relevant amount of heterogeneity and thus the dynamics of the model.

Accuracy tests. As an additional validation test, we perform two other standard accuracy tests. We start with Euler Equation error tests (see Den Haan and Marcet, 1994, Aruoba et al. 2006, and Den Haan, 2010) on both the steady-state model and the model with aggregate shocks. They consist in computing the absolute errors (on a base-10 log scale) implied by the exact Euler equations using the simulated allocations. The results (including mean, standard deviation, and distribution of errors) are reported in Table 6. A value of -3 for this error, which is approximately the value found in the three cases,

	Static	Dynamic	
	Bewley	Exo. ϕ	Endo. ϕ
Average	-3.91	-4.08	-3.95
Std. dev.	0.63	0.57	0.70
$[-2, \infty)$	0.00	0.00	0.00
$[-3, -2)$	1.47	3.01	5.11
$[-4, -3)$	96.57	32.14	53.93
$[-5, -4)$	1.78	59.72	33.73
$[-6, -5)$	0.06	4.59	5.94
$(-\infty, -6)$	0.10	0.53	1.29

Table 6: Euler equation errors

means a \$1 mistake for \$1,000 of consumption. This is generally considered as being an acceptable error, as discussed in Faraglia et al. (2019) among others.

For the steady-state model, the results can be found in the column of Table 6 labeled Bewley. For the benchmark economy with $\phi = 42\%$, the average error amounts to -3.91, which lies within the admissible range. The results for the model with aggregate shocks are reported in the third and fourth rows of Table 6. This accuracy test is all the more important since we use a perturbation approach, which ignores some potential non-linearities. We consider two cases for the replacement rate: an exogenous ϕ, set to 42%, (third column) and an optimal time-varying rate (fourth column). The mean absolute log error is -4.08 in the exogenous case and -3.95 in the endogenous one, which are in the admissible range.

Our second set of tests concerns the assumption that credit-constrained histories remain constant in the dynamics (second point of Assumption A). The test is run as follows. We simulate the model – with both exogenous and endogenous replacement rates – over $10,000$ periods. We then check that the saving decision of each unconstrained history remains above the credit limit in the simulations. Conversely, we also check that credit-constrained histories remain constrained in the dynamics by checking the sign of the Euler inequality.

We find that savings remain positive for all unconstrained histories and that all Euler inequalities have the correct sign for constrained histories. Finally, as noted by Den Haan (2010), these accuracy tests are not sufficient to characterize the overall goodness-of-fit of a simulation method. We consider the results of Table 4 as the main evidence of the truncation method's relevance.

6 Conclusion

This paper presents a truncation representation of incomplete insurance market models with aggregate shocks. We construct a finite-dimensional state-space representation, which can be simulated with aggregate shocks, and for which optimal Ramsey policies can be derived. We apply the theory to characterize optimal time-varying unemployment benefits when the economy is hit by aggregate shocks. The model simulation uses perturbation methods, which considerably eases implementation. Such methods, however, rely on small aggregate shocks around a well-defined steady state. They are less relevant for models with large macroeconomic shocks, for which additional developments are needed, using penalty functions or global methods.

The theory could obviously be used for many other applications. The underlying model could be generalized to examine relevant frictions on the goods, labor, or financial markets, such as limited participation on financial markets or nominal frictions. In addition, the planner could use other tools to reduce distortions, such as a whole set of fiscal or monetary policy instruments. We are currently working on the design of general optimal fiscal policies in these environments. The simplicity of the implementation could contribute to a more systematic integration of redistributive effects in the design of economic policies.

Appendix

This Appendix is split into five parts. Section A offers a micro-foundation theory to the truncation method. Section B contains the proofs of the paper. Section C presents the details of the truncation method. Section D contains a number of robustness checks and additional results of our numerical implementation. Finally, Section E presents supplemental theoretical results.

A Decentralizing the truncation method

In this section, we explain how the truncation method of Section 4 can be seen as the outcome of a decentralized mechanism. More precisely, we show that the truncated allocation can be seen as the market outcome of an island economy (Section A.1) – see Lucas, 1975, 1990, or Heathcote et al., 2017 for a more recent reference. We also prove that the island economy can be decentralized by a well-chosen fiscal system of lump-sum transfers (A.2). We denote the truncation length by $N \geq 0$.

A.1 The island metaphor

Island description. There are S^N different islands, where S is the cardinal of $\mathcal{S}$. Agents with the same idiosyncratic history for the last N periods are located on the same island. Any island is represented by a vector $s^N = (s^N_{-N+1}, \ldots, s^N_0) \in \mathcal{S}^N$ summarizing the last N-period idiosyncratic history of all island inhabitants. At the beginning of each period, agents face a new idiosyncratic shock. Agents with history $\hat{s}^N$ in the previous period are endowed with the new history s^N in the current period. The history s^N will be said to be a continuation of $\hat{s}^N$, and will be denoted by $s^N \succeq \hat{s}^N$. The probability of transitioning from island $\hat{s}^N$ at $t-1$ to island s^N at date t is denoted by $\Pi_{t,\hat{s}^N,s^N}$, defined in (20). The island sizes $(S_{t,s^N})_{s^N \in \mathcal{S}^N}$ can be deduced from these probabilities and are defined in recursion (21).

The specification $N = 0$ (one island) corresponds to the standard representative-agent model. Symmetrically, the case $N = \infty$ corresponds to a standard incomplete-market economy with aggregate shocks, as in Krusell and Smith (1998).

The quasi-planner. The quasi-planner maximizes the welfare of agents, attributing an identical weight to all agents and behaving as a price-taker.[19] The quasi-planner can freely transfer resources among agents on the same island, but cannot do so across islands. All agents belonging to the same island are treated identically and therefore receive the same allocation, as is consistent with welfare maximization. For island s^N, the quasi-planner

[19]As the quasi-planner does not internalize the effect of its choices on prices, the allocation is not constrained-efficient, and the distortions identified by Dávila et al. (2012) are present in the equilibrium allocation. Section 3 introduces a Ramsey planner that will choose an UI policy to reduce these distortions.

will choose the per capita consumption level c_{t,s^N}, the labor supply l_{t,s^N}, and the savings a_{t,s^N}.

Wealth pooling and heterogeneity reduction. At the beginning of each period t, agents learn about their current idiosyncratic shock and moves from island $\hat{s}^N$ to island s^N. Agents take their wealth – equal to $a_{t-1,\hat{s}^N}$ – with them when they move. On island s^N, the wealth of all agents coming from island $\hat{s}^N$ (equal to $S_{t-1,\hat{s}^N}\Pi_{t,\hat{s}^N,s^N}a_{t-1,\hat{s}^N}$) – and for all islands $\hat{s}^N$– is pooled together and then equally divided among the S_{t,s^N} agents of island s^N. Therefore, at the *beginning of period t*, each agent of s^N holds wealth $\tilde{a}_{t,s^N}$, defined in equation (23). We denote by $(a_{-1,s^N})_{s^N\in\mathcal{S}^N}$ the initial wealth endowment.

As explained above, agents face island-specific preference shifters, denoted by ξ_{s^N}, that multiply their utility function. The quasi-planner's program can be expressed as:

$$\max_{(c_{t,s^N},l_{t,s^N},a_{t,s^N},\tilde{a}_{t,s^N})_{t\geq 0,s^N\in\mathcal{S}^N}} \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{s^N\in\mathcal{S}^N} S_{t,s^N}\xi_{s^N}U(c_{t,s^N},\hat{l}_{t,s^N}), \tag{35}$$

$$a_{t,s^N}+c_{t,s^N}=((1-\tau_t)l_{t,s^N}1_{e_0^N=e}+\phi_t l_{t,s^N,e}1_{e_0^N=u})y_0^N w_t+(1+r_t)\tilde{a}_{t,s^N}\ (s^N\in\mathcal{S}^N), \tag{36}$$

$$c_{t,s^N},\ l_{t,s^N}\geq 0, a_{t,s^N}\geq -\bar{a}\ (s^N\in\mathcal{S}^N), \tag{37}$$

and subject to the law of motion for $(S_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$, and to the definition of $(\tilde{a}_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$. Note that initial island sizes $(S_{-1,s^N})_{s^N\in\mathcal{S}^N}$ and initial wealth $(a_{-1,s^N})_{s^N\in\mathcal{S}^N}$ are given.[20]

As the objective function is increasing and concave, constraints are linear (i.e., the admissible set is convex), and the existence of the equilibrium can be proved using standard techniques (see Stokey et al., 1989, Chap. 15 and 16). We therefore omit this proof in the interest of conciseness.

The first-order conditions of the quasi-planner's program are the same as those derived in the main text for the truncated economy (same notation):

$$\xi_{s^N}U_c(c_{t,s^N},\hat{l}_{t,s^N})=\beta\mathbb{E}_t\Big[\sum_{\tilde{s}^N\succeq s^N}\Pi_{t+1,s^N,\tilde{s}^N}\xi_{\tilde{s}^N}U_c(c_{t+1,\tilde{s}^N},\hat{l}_{t+1,\tilde{s}^N})(1+r_{t+1})\Big]+\nu_{t,s^N}, \tag{38}$$

$$l_{t,s^N}^{1/\varphi}=\chi(1-\tau_t)w_t y_0^N 1_{e_0^N=e}, \tag{39}$$

$$\nu_{t,s^N}(a_{t,s^N}+\bar{a})=0 \text{ and } \nu_{t,s^N}\geq 0. \tag{40}$$

Market clearing and equilibrium. The clearing for labor and capital markets implies equations (28). We can now state our sequential equilibrium definition, which is similar to the definition of the truncated equilibrium in the main text (Definition 2).

Definition 3 (Sequential equilibrium) *A sequential truncated competitive equilibrium is a collection of individual allocations* $(c_{t,s^N},l_{t,s^N},a_{t,s^N},\tilde{a}_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$*, of island population*

[20]In equation (36), $l_{t,s^N,e}$ is the labor supply of an employed agent with productivity y_0^N, which determines the UI benefits of unemployed agents of history s^N. Furthermore, as in (10), $\hat{l}_{t,s^N}=l_{t,s^N}1_{e_0^N=e}+\zeta_{y_0^N}1_{e_0^N=u}$.

sizes $(S_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$*, of aggregate quantities* $(L_t, K_t, Y_t)_{t\geq 0}$*, of price processes* $(w_t, r_t)_{t\geq 0}$*, and of UI policy* $(\tau_t, \phi_t)_{t\geq 0}$*, such that, for an initial distribution of island population and wealth* $(S_{-1,s^N}, a_{-1,s^N})_{s^N\in\mathcal{S}^N}$*, and for initial values of capital stock* K_{-1} *and of the initial aggregate shock* z_{-1}*, we have:*

1. *given prices, individual strategies* $(c_{t,s^N}, l_{t,s^N}, a_{t,s^N}, \tilde{a}_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$ *solve the agents' optimization program in equations (35)–(37);*
2. *island sizes and beginning-of-period individual wealth* $(S_{t,s^N}, \tilde{a}_{t,s^N})_{t\geq 0}^{s^N\in\mathcal{S}^N}$ *are consistent with the laws of motion (21) and (23);*
3. *capital and labor markets clear at all dates: for any* $t \geq 0$*, equations (28) hold;*
4. *the UI budget is balanced at all dates: equation (14) holds for all* $t \geq 0$*;*
5. *factor prices* $(w_t, r_t)_{t\geq 0}$ *are consistent with the firm's program (3).*

The sequential equilibrium has a simple structure defined at each date by $5S^N + 5$ variables and $5S^N + 5$ equations for a given UI policy $(\tau_t, \phi_t)_{t\geq 0}$. The equilibrium features a finite number of different allocations, characterized by the N-period history of agents.

A.2 A decentralization mechanism

We now prove that the finite-state equilibrium of Definition 3 can be decentralized through fiscal transfers, which are shown to measure the degree of idiosyncratic risk sharing achieved by asset pooling in the island economy. The economy is now similar to that in Section 2 – in particular, agents are expected-utility maximizers – except for two differences. First, agents are endowed with so-called *preference shifters*. Second, at each date, each agent receives a lump-sum transfer Γ_{N+1}, which is contingent on her individual history s^{N+1} over the previous $N + 1$ periods. This fiscal system will be key in mimicking the pooling operation of Section A.1. Using standard recursive notation, the agents' program can be written as:[21]

$$V(a, s^{N+1}, X) = \max_{a',c,l} \; \xi_{s^N} U(c, \hat{l}) + \beta \mathbb{E}\Big[\sum_{(s^{N+1})'} \Pi'_{s^{N+1},(s^{N+1})'} V(a', (s^{N+1})', X')\Big], \quad (41)$$

$$a' + c - (1 + r(X))a = ((1 - \tau)l 1_{e_0^N = e} + \phi l_{y,e} 1_{e_0^N = u}) w(X) + \Gamma_{N+1}(s^{N+1}, X), \quad (42)$$

$$\hat{l} = l 1_{e_0^N = e} + \zeta_{y_0^N} 1_{e_0^N = u}, \quad (43)$$

$$c, \, l \geq 0, \; a' \geq -\bar{a}, \quad (44)$$

where $l_{y,e}$ denotes the labor supply of an employed agent with productivity y, and where the state vector X encompasses all variables necessary to forecast prices, including aggregate shocks. Compared to the economies studied by Huggett (1993) and Aiyagari (1994), the individual history s^{N+1} is a state variable, as it determines the transfer $\Gamma_{N+1}(s^{N+1}, X)$.

We now state our result, which explains that we can find a particular set of transfers – denoted by $(\Gamma^*_{N+1}(s^{N+1}, X))_{s^{N+1}\in\mathcal{S}^{N+1}}$ – such that the decentralized economy allocations match those of the island economy.

[21] As standard, we denote the current savings choice by a'; a is thus the beginning-of-period wealth.

Proposition 5 (Finite state space) *A set of balanced transfers exists, that are denoted by* $(\Gamma^*_{N+1}(s^{N+1}, X))_{s^{N+1}\in\mathcal{S}^{N+1}}$*, such that any optimal allocation of the island program (35)–(37) is also a solution to the decentralized program (41)–(44).*

Proposition 5, proved in Section A.3, states that the island program presented in Section A.1 can be decentralized by the balanced lump-sum transfers $(\Gamma^*_{N+1}(s^{N+1}, X))_{s^{N+1}\in\mathcal{S}^{N+1}}$ (shortened to Γ^*_{N+1} henceforth). This transfer is formally provided in equation (46) of Appendix A.3.

The transfers Γ^*_{N+1} mimic the wealth pooling of the island economy (equation (23)), when agents transfer from one island to another. It consists of two steps: (i) putting together the beginning-of-period wealth of all agents with the same idiosyncratic history for the last N periods, independently of their idiosyncratic status $N+1$ periods ago, and (ii) redistributing consistently the same amount to agents with the same idiosyncratic history for N periods, such that there are only S^N possible wealth levels. The transfers (Γ^*_{N+1}) operate a strict redistribution among agents and thus sum up to zero.

A.3 Proof of Proposition 5

We use a guess-and-verify strategy. The transfer is constructed such that all agents with the same N-period history have the same after-transfer wealth. The measure of agents with history s^N follows the same law of motion as (21) in the main text and is equal to S_{s^N}. If agents with the same history $(\hat{s}^N, s), s \in \mathcal{S}$ have the same beginning-of-period wealth $a_{\hat{s}^N}$, the after-transfer wealth, denoted by $\hat{a}_{s^N}$, of agents with history $s^N \succeq \hat{s}^N$ is:

$$\hat{a}'_{s^N} = \sum_{\tilde{s}^N\in\mathcal{S}^N} \frac{S_{\tilde{s}^N}}{S_{s^N}} \Pi_{\tilde{s}^N, s^N} a'_{\tilde{s}^N}, \tag{45}$$

such that agents with the same history hold the same wealth. By construction, $\hat{a}_{s^N}$ follows dynamics similar to the "after-pooling" wealth $\tilde{a}_{t,s^N}$ in the island economy of equation (23). The transfer scheme denoted by $(\Gamma^*_{N+1}(s^{N+1}, X))_{s^{N+1}\in\mathcal{S}^{N+1}}$ that enables all agents with the same history to have the same wealth is:

$$\Gamma^*_{N+1}(s^{N+1}, X) = (1+r)\left(\hat{a}_{s^N} - a_{\hat{s}^N}\right), \tag{46}$$

where we use $s^{N+1} = (\hat{s}^N, s) = (s_N, s^N)$ (in the former notation, s^{N+1} is seen as the history $\hat{s}^N \in \mathcal{S}^N$ with the successor state $s \in \mathcal{S}$, while in the latter notation, s^{N+1} is seen as the state $s_N \in \mathcal{S}$ followed by history $s^N \in \mathcal{S}^N$). The transfer Γ^*_{N+1} defined in (46) replaces the beginning-of-period wealth $(1+r)a_{\hat{s}^N}$ with the *average* wealth $(1+r)\hat{a}_{s^N}$, which only depends on the last N-period history. Since there is a continuum with mass $S_{\tilde{s}^N}$ of agents with history $\tilde{s}^N$, in which each individual agent is atomistic, all agents take the transfer Γ^*_{N+1} as given.

Finally, it is easy to check that the transfer scheme is balanced in each period. Using the definition (45) of $\hat{a}_{s^N}$, we obtain for $s^N = (s^N_{N-1}, \ldots, s^N_1, s^N_0) \in \mathcal{S}^N$, $S_{s^N}\hat{a}_{s^N} =$

$\sum_{\hat{s}^N \in \mathcal{S}^N} S_{\hat{s}^N} \Pi_{\hat{s}^N, s^N} a_{\hat{s}^N} = \sum_{\hat{s} \in \mathcal{S}} S_{(\hat{s}, s^N_{N-1}, \ldots, s^N_1)} M_{s^N_1, s^N_0} a_{(\hat{s}, s^N_{N-1}, \ldots, s^N_1)}$. Therefore, we deduce that: $\sum_{\tilde{s} \in \mathcal{S}} S_{(\tilde{s}, s^N)} \Gamma^*_{N+1}(\tilde{s}, s^N) = (1+r) \left[\sum_{\tilde{s} \in \mathcal{S}} S_{(\tilde{s}, s^N)} \left(\hat{a}_{s^N} - a_{(\tilde{s}, s^N_{N-1}, \ldots, s^N_1)} \right) \right] = 0$, where the last equality comes from the definition of $\hat{a}_{s^N}$ in equation (45).

B Proofs

B.1 The full-fledged Ramsey program

Rewriting the Ramsey program. Let $\beta^t \lambda^i_t$ be the Lagrange multiplier on the Euler equation (8). Including the Euler equation constraint in the planner's objective (15) yields, using $\nu^i_t \lambda^i_t = 0$, the following objective, denoted by J:

$$J = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \int_i \left(U^i_t \ell(di) - (\lambda^i_t - \lambda^i_{t-1}(1+r_t)) U^i_{c,t} \right) \ell(di), \tag{47}$$

where we use the following notation: $U^i_t = U(c^i_t, \hat{l}^i_t)$, $U^i_{c,t} = U_{c,t}(c^i_t, \hat{l}^i_t)$ (and similarly for $U^i_{l,t}$, $U^i_{cc,t}$, and $U^i_{cc,t}$). The Ramsey program then consists in maximizing J in (47) over $((a^i_t, c^i_t, l^i_t)_i, \phi_t, \tau_t)_{t \geq 0}$ subject to the relevant constraints: equations (6), (14), and (28). Using (3) to substitute for r_t and w_t, the Lagrangian can be seen as depending only on (a^i_t) and ϕ_t.

FOC with respect to saving choices a^i_t. Using equations (11), we can show for aggregate quantities that $\frac{\partial K_{t-1}}{\partial a^i_t} = \frac{\partial L_t}{\partial a^i_t} = 0$, and:

$$\frac{\partial K_t}{\partial a^i_t} = 1, \ \frac{\partial L_{t+1}}{\partial a^i_t} = \frac{\varphi L_{t+1} \frac{F_{KL,t+1}}{F_{L,t+1}}}{1 - \varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}}, \ \frac{\partial w_{t+1}}{\partial a^i_t} = \frac{F_{KL,t+1}}{1 - \varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}}.$$

After some manipulations, and using $F(K,L) = K^\alpha L^{1-\alpha} - \delta K$, we obtain for c^i_t and l^i_t:

$$\begin{aligned}
&\frac{\partial l^i_t}{\partial a^j_t} = 0, \ \frac{\partial l^i_{t+1}}{\partial a^j_t} = \varphi \frac{\alpha K_t^{-1}}{1+\alpha\varphi} l^i_{t+1} 1_{e^i_t = e}, \frac{\partial c^i_t}{\partial a^j_t} = -1_{j=i}, \\
&\frac{\partial c^i_{t+1}}{\partial a^j_t} = (1+r_{t+1}) 1_{i=j} + \frac{F_{KK,t+1}}{1+\alpha\varphi} a^i_{t-1} \\
&\quad + \left((1-\tau_{t+1}) 1_{e^i_{t+1}=e} + \frac{S_{t+1,e}}{S_{t+1,u}} \tau_{t+1} 1_{e^i_{t+1}=u} \right) l^i_{t+1,e} y^i_{t+1} (1+\varphi) \frac{\alpha w_t K_t^{-1}}{1+\alpha\varphi}.
\end{aligned}$$

The derivative of the Lagrangian with respect to a^j_t implies, using (16):

$$\begin{aligned}
\psi^j_t = {} & \beta \mathbb{E}_t \left[\int_i \psi^i_{t+1} \left(\frac{\partial c^i_{t+1}}{\partial a^i_t} - \chi^{-1} l^{i,\frac{1}{\varphi}}_{t+1} \frac{\partial l^i_{t+1}}{\partial a^i_t} \right) \ell(di) \right] \\
& + \beta \mathbb{E}_t \left[\int_i \left(F_{KK,t+1} + F_{KL,t+1} \frac{\varphi L_{t+1} \frac{F_{KL,t+1}}{F_{L,t+1}}}{1 - \varphi L_{t+1} \frac{F_{LL,t+1}}{F_{L,t+1}}} \right) \lambda^i_t U^i_{c,t+1} \ell(di) \right].
\end{aligned} \tag{48}$$

With $\frac{F_{KL}^2 - F_{KK}F_{LL}}{F_L} = 0$ and the expression of $\frac{\partial c_{t+1}^i}{\partial a_t^i} - \chi^{-1} l_{t+1}^{i,\frac{1}{\varphi}} \frac{\partial l_{t+1}^i}{\partial a_t^i}$, we obtain FOC (17).

FOC with respect to replacement rate ϕ_t. Rather than computing the derivative with respect to ϕ_t, we do so with respect to τ_t. For aggregate quantities, we obtain quite directly: $\frac{\partial L_t}{\partial \tau_t} = -\frac{\varphi}{1+\alpha\varphi}\frac{L_t}{1-\tau_t}$, $\frac{\partial K_{t-1}}{\partial \tau_t} = 0$. The computation for individual choices is lengthier, and yields:

$$\frac{\partial l_t^i}{\partial \tau_t} = -\frac{\frac{\varphi}{1-\tau_t} l_t^i}{1+\alpha\varphi} 1_{e_t^i = e},$$
$$\frac{\partial c_t^i}{\partial \tau_t} = -\frac{\alpha\varphi \frac{w_t}{1-\tau_t}\frac{L_t}{K_{t-1}}}{1+\alpha\varphi} a_{t-1}^i + \frac{S_{t,e}}{S_{t,u}} \frac{l_t^i}{1-\tau_t} y_t^i w_t 1_{e_t^i = u} \left(1 - \tau_t \frac{1+\varphi}{1+\alpha\varphi}\right) - 1_{e_t^i = e} l_t^i y_t^i w_t \frac{1+\varphi}{1+\alpha\varphi}.$$

Using the Lagrangian expression, we obtain after substituting for ψ_t^i and partial derivatives:

$$0 = \int_i \psi_t^i \left(\frac{S_{t,e}}{S_{t,u}} \frac{l_t^i}{1-\tau_t} y_t^i w_t \frac{1-\tau_t-\varphi(1-\alpha)}{1+\alpha\varphi} 1_{e_t^i = u} - 1_{e_t^i = e} l_t^i y_t^i w_t \frac{1}{1+\alpha\varphi} \right) \ell(di)$$
$$- \frac{\alpha\varphi}{1+\alpha\varphi} \frac{w_t}{1-\tau_t} \frac{L_t}{K_{t-1}} \int_i (\lambda_{t-1}^i U_{c,t}^i + \psi_t^i a_{t-1}^i) \ell(di),$$

which yields equation (18) after some rearrangement.

B.2 The Ramsey program in the truncated economy

The computations are similar to those in the full-fledged Ramsey program (Section B.1).

B.2.1 Rewriting the Ramsey program

In the remainder, we use the notation $U_{c,t,s^N} = u'(c_{t,s^N} - \chi^{-1} \frac{\hat{l}_{t,s^N}^{1+1/\varphi}}{1+1/\varphi})$, and similarly for U_{cc,t,s^N}, U_{cl,t,s^N}, and U_{ll,t,s^N}. The planner's program can be written as:

$$\max_{\left((a_{t,s^N}, c_{t,s^N}, l_{t,s^N})_{s^N}, \phi_t, \tau_t, r_t, w_t\right)_{t\geq 0}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \xi_{t,s^N} U(c_{t,s^N}, \hat{l}_{t,s^N}) \right], \tag{49}$$
$$\xi_{t,s^N} U_{c,t,s^N} - \nu_{t,s^N} = \beta \mathbb{E}_t \Big[(1+r_{t+1}) \sum_{\tilde{s}^N \in \mathcal{S}^N} \Pi_{t+1,s^N \tilde{s}^N} \xi_{t+1,\tilde{s}^N} U_{c,t+1,s^N} \Big], \tag{50}$$

and subject to (24), (28), (14), and (3). Let $\beta^t S_{t,s^N} \lambda_{t,s^N}$ be the Lagrange multiplier on the Euler equation. With $\nu_{t,s^N} \lambda_{t,s^N} = 0$, the planner's objective, denoted by J, becomes:

$$J = \mathbb{E}_0 \sum_{t,s^N} \beta^t \Big(S_{t,s^N} \xi_{t,s^N} U(c_{t,s^N}, \hat{l}_{t,s^N}) - (\lambda_{t,s^N} - (1+r_t)\tilde{\lambda}_{t,s^N}) \xi_{t,s^N} U_c(c_{t,s^N}, \hat{l}_{t,s^N}) \Big). \tag{51}$$

The Ramsey program consists in maximizing J in (51) over $((a_{t,s^N}, c_{t,s^N}, l_{t,s^N})_{s^N}, \phi_t, \tau_t)_{t\geq 0}$ subject to the relevant constraints. As in the full-fledged case, the Lagrangian can be seen as depending only on saving choices (a_{t,s^N}) and the replacement rate ϕ_t.

FOC with respect to saving choices a_{t,s^N} Using equations (28), we can show for aggregate quantities that:

$$\frac{\partial K_t}{\partial a_{t,s^N}} = S_{t,s^N}, \; \frac{\partial K_{t-1}}{\partial a_{t,s^N}} = 0, \; \frac{\partial L_{t+1}}{\partial a_{t,s^N}} = \frac{\varphi L_{t+1}\frac{F_{KL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N}, \; \frac{\partial L_t}{\partial a_{t,s^N}} = 0.$$

After some manipulations, we obtain for individual choices (labor and consumption):

$$\frac{\partial l_{t,\tilde{s}^N}}{\partial a_{t,s^N}} = 0, \; \frac{\partial l_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}} = \varphi \frac{\frac{F_{KL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N} l_{\tilde{s}^N,e,t+1} 1_{e_{\tilde{s}^N}=e}, \; \frac{\partial c_{t,\tilde{s}^N}}{\partial a_{t,s^N}} = -1_{s^N=\tilde{s}^N},$$

$$\begin{aligned}\frac{\partial c_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}} &= (1+F_{K,t+1})\Pi_{t+1,s^N\tilde{s}^N} + \frac{F_{KK,t+1}+\varphi L_{t+1}\frac{F^2_{KL,t+1}-F_{KK,t+1}F_{LL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N}\tilde{a}_{t,s^N} \\ &+ \left((1-\tau_{t+1})1_{e_{\tilde{s}^N}=e} + \frac{S_{t+1,e}}{S_{t+1,u}}\tau_{t+1}1_{e_{\tilde{s}^N}=u}\right) l_{\tilde{s}^N,e,t+1}\tilde{y}\,(1+\varphi)\frac{F_{KL,t+1}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N}.\end{aligned}$$

We deduce for $\frac{\partial C_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}} = \frac{\partial c_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}} - \chi^{-1} l_{t+1,\tilde{s}^N}^{\frac{1}{\varphi}} \frac{\partial l_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}}$:

$$\begin{aligned}\frac{\partial C_{t+1,\tilde{s}^N}}{\partial a_{t,s^N}} &= (1+r_{t+1})\Pi_{t+1,s^N\tilde{s}^N} + \frac{F_{KK,t+1}+\varphi L_{t+1}\frac{F^2_{KL,t+1}-F_{KK,t+1}F_{LL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N}\tilde{a}_{t,s^N} \\ &+ \left((1-\tau_{t+1})1_{e_{\tilde{s}^N}=e} + \frac{S_{t+1,e}}{S_{t+1,u}}\tau_{t+1}1_{e_{\tilde{s}^N}=u}\right) l_{\tilde{s}^N,e,t+1}\tilde{y}\,(1+\varphi)\frac{F_{KL,t+1}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N} \\ &-(1-\tau_{t+1}) y_{\tilde{s}^N}\varphi \frac{F_{KL,t+1}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}} S_{t,s^N} l_{\tilde{s}^N,e,t+1}\end{aligned} \quad (52)$$

Note that using $F(K,L) = K^\alpha L^{1-\alpha} - \delta K$, we have: $\frac{F^2_{KL}-F_{KK}F_{LL}}{F_L} = 0$; $L\frac{F_{LL}}{F_L} = -\alpha$; $F_{KL} = F_L\frac{\alpha}{K_t}$ and $F_{KK} = -\alpha F_L \frac{L}{K^2}$. Simplifying the partial derivatives of F and using (52), the derivative of the Lagrangian with respect to a_{t,s^N} implies equation (32).

FOC with respect to ϕ_t Rather than computing the derivative with respect to ϕ_t, we do so with respect to τ_t. For aggregate quantities, we obtain quite directly: $\frac{\partial L_t}{\partial \tau_t} = -\frac{\varphi\frac{L_t}{1-\tau_t}}{1-\varphi L_t\frac{F_{LL,t}}{F_{L,t}}}$, $\frac{\partial K_{t-1}}{\partial \tau_t} = 0$. The computation for individual choices is lengthier, and yields:

$$\begin{aligned}\frac{\partial l_{t,\tilde{s}^N}}{\partial \tau_t} &= -\frac{\frac{\varphi}{1-\tau_t} l_{t,\tilde{s}^N}}{1-\varphi L_t\frac{F_{LL,t}}{F_{L,t}}} 1_{e_{\tilde{s}^N}=e}, \\ \frac{\partial c_{t,\tilde{s}^N}}{\partial \tau_t} &= -\frac{\varphi\frac{L_t}{1-\tau_t}}{1-\varphi L_t\frac{F_{LL,t}}{F_{L,t}}} F_{KL,t}\tilde{a}_{t,\tilde{s}^N} + (1-\tau_t)^{\varphi-1}\frac{S_{t,e}}{S_{t,u}}\chi^\varphi \tilde{y}^{\varphi+1} F_{L,t}^{1+\varphi} 1_{e_{\tilde{s}^N}=u} \\ &\quad -\left((1-\tau_t)1_{e_{\tilde{s}^N}=e} + \frac{S_{t,e}}{S_{t,u}}\tau_t 1_{e_{\tilde{s}^N}=u}\right)\chi^\varphi \tilde{y}^{\varphi+1}(1+\varphi)\frac{(1-\tau_t)^{\varphi-1}F_{L,t}^{1+\varphi}}{1-\varphi L_t\frac{F_{LL,t}}{F_{L,t}}}.\end{aligned}$$

Using the partial derivatives of F and previous computations, we obtain equation (33).

B.3 Proof of the convergence of the truncated equilibrium

The proof is performed is several steps: (i) we first show the convergence of allocations in the absence of aggregate shocks; (ii) we then show that the $(\xi_{s^N})_{s^N}$ converge to 1; and (iii) we finally prove convergence in the presence of aggregate shocks in the context of a resolution via perturbation method.

B.3.1 Proof of the first part of Proposition 2

In this proof, there is no aggregate shock. The starting point of this proof is that the allocations of the aggregated model are computed using the aggregation of Bewley allocations. From equation (22) for the truncation of a variable X, it can be seen that X_{t,s^N} is the integral of X over some specific set, which is defined as the histories whose last N periods match the vector s^N. As N gets larger, the sum is defined over an increasingly small set. Loosely speaking, a generalized version of the fundamental theorem of calculus then guarantees the convergence.

To formalize this loose statement, we need to introduce some notation. Let $s^\infty = (s_0, s_1, \ldots)$ be an infinite idiosyncratic history, which can also be seen as a Markov chain where each s_t belongs to $(\mathcal{S}, \mathcal{F}_\mathcal{S})$, where $\mathcal{F}_\mathcal{S}$ are the σ-algebras generated by $\mathcal{S}$ (which is finite in our truncated case). The whole history s^∞ lies in the sequence space $\Omega = \mathcal{S} \times \mathcal{S} \times \ldots = \mathcal{S}^\infty$ endowed with the product σ-algebra $\mathcal{F}_\infty = \mathcal{F}_\mathcal{S}^\infty$ and the measure μ_∞. We recall that the set Ω is uncountably infinite and has the cardinality of the continuum. The measure μ_∞ exists and is defined as the infinite product measure that coincides with the standard Markov distribution for any finite sequence. This measure can be shown to be the limit of the product measure of transition kernels (generalizing the transition matrix in the case of a non-finite state space $\mathcal{S}$). An important feature of the measure μ_∞ is that it is consistent with the usual Markov measure for any finite sequence. The proof of the existence of an infinite product measure is in general quite involved and relies on the Kolmogorov extension theorem (see Tao, 2011, Theorem 2.4.3). In our case of a finite Markov chain, the infinite measure is uniquely determined by its initial distribution and its transition matrix (see Brémaud, 2014, Theorem 1.1).

We now consider the probability space $(\Omega, \mathcal{F}, \mu_\infty)$. We denote by $\mathcal{S}_N$ the partition of idiosyncratic histories, induced by the truncation: an idiosyncratic history is uniquely defined by its truncation over the last N periods. Two infinite histories with the same last N idiosyncratic realization belong to the same truncated history. A truncated history $s_N \in \mathcal{S}_N$ can then be seen as a subset of Ω and the truncation of the variable X in equation (22) can be written as: $X_{t,s^N} = \int_{s^\infty \in s_N} X_t(s^\infty) \frac{\mu_\infty(ds^\infty)}{\int_{s^\infty \in s_N} \mu_\infty(ds^\infty)}$.

We denote by $\mathcal{F}_N$ the filtration associated with the partition $\mathcal{S}_N$. The conditional probability $\mathbb{E}\left[X_t | \mathcal{F}_N\right]$ – which is a random variable – verifies for any event $s^\infty \in \Omega$: $\mathbb{E}\left[X_t | \mathcal{F}_N\right]_{s^\infty} = X_{t,s^N}$, where s^N is the label of the unique partition element s_N containing

s^∞. In other words, the restriction of the conditional expectation to h_n coincides with X_{t,s^N}. We can state two additional properties on the filtration sequence $(\mathcal{F}_N)_{N\geq 0}$.

1. The partition sequence $(\mathcal{S}_N)_{N\geq 0}$ is increasing by construction and is such that $\mathcal{S}_{N+1}$ is a refinement of $\mathcal{S}_N$, in the sense that any element of $\mathcal{S}_N$ is a union of elements of $\mathcal{S}_{N+1}$. The filtration sequence $(\mathcal{F}_N)_{N\geq 0}$ is thus increasing: $\mathcal{F}_N \subset \mathcal{F}_{N+1}$.

2. The partition sequence $(\mathcal{S}_N)_{N\geq 0}$ converges to the atoms of Ω – in other words, histories become infinitely long – which implies $(\mathcal{F}_N) \uparrow \mathcal{F}_\infty$.

To conclude the convergence proof, we apply the convergence theorem for conditional expectation (see Billingsley, 1965, Theorem 11.2), which yields: $\mathbb{E}\left[X|\mathcal{F}_N\right]_{s^\infty} \rightarrow_{N\to\infty} X(s^\infty)$, almost surely. This concludes the proof of the first part of Proposition 2.

B.3.2 Proof of the second part of Proposition 2

Let $(\mathcal{S}_N)_{N\geq 0}$ be an increasing partition sequence associated with history truncation, as described in Section B.3.1. We will define $\xi^u_{s^N} = \frac{\mathbb{E}\left[U_c(c_{s^N}, \hat{l}_{s^N})|\mathcal{F}_N\right]_{s^\infty}}{U_c(\mathbb{E}\left[c_{s^N}|\mathcal{F}_n\right]_{s^\infty}, \mathbb{E}\left[\hat{l}_{s^N}|\mathcal{F}_n\right]_{s^\infty})}$ and $\Pi^u_{t+1,\tilde{s}^N s^N} = \Pi_{t+1,\tilde{s}^N s^N} \frac{\sum_{s^t\in\mathcal{S}^t|(s^t_{t-N+1},\ldots,s^t_t)=\tilde{s}^N} U_{c,t+1,(s^t,s^N_0)} \frac{\mu_t(s^t)}{S_{t,\tilde{s}^N}}}{U_{c,t+1,s^N}}$ (where $U_{c,t+1,s^N} = U_c(c_{t+1,\tilde{s}^N}, \hat{l}_{t+1,\tilde{s}^N}) > 0$ for all s^N). We then check that truncating the individual Euler equation (8) yields:

$$\xi^u_{s^N} U_{c,s^N} = \nu_{s^N} + \beta(1+r) \sum_{\tilde{s}^N\in\mathcal{S}^N} \Pi^u_{s^N\tilde{s}^N} \xi^u_{\tilde{s}^N} U_{c,\tilde{s}^N}, \tag{53}$$

which is similar to the aggregated Euler equation (25), except that ξs and transition probabilities are different (these ξ^u and probabilities Π^u are actually constructed for this equality to be exactly true). The convergence result of Section B.3.1 and the continuity of U_c imply that $\xi^u_{t,s^N} \to \frac{U_c(c(s^\infty),\hat{l}(s^\infty))}{U_c(c(s^\infty),\hat{l}(s^\infty))} = 1$, almost surely. Similarly, we have:

$$U_{c,t+1,\tilde{s}^N}(\Pi^u_{t+1,s^N\tilde{s}^N} - \Pi_{t+1,s^N\tilde{s}^N}) = \Pi_{t+1,\tilde{s}^N s^N} \sum_{(s^t_{t-N+1},\ldots,s^t_t)=s^N} (U_{c,t+1,(s^t,s^N_0)} - U_{c,t+1,\tilde{s}^N}) \frac{\mu_t(s^t)}{S_{t,\tilde{s}^N}},$$

which, using Section B.3.1 and the continuity of U_c, can be shown to converge to 0.

We consider the steady-state difference between equation (53) and the truncated (island) consumption Euler equation for a history $s^N \in \mathcal{S}^N$. After some manipulation, we obtain:

$$\begin{aligned}(\xi^u_{s^N} - \xi_{s^N})U_{c,s^N} = {} & \beta(1+r) \sum_{\tilde{s}^N\in\mathcal{S}^N} \Pi_{s^N\tilde{s}^N}(\xi^u_{\tilde{s}^N} - \xi_{\tilde{s}^N})U_{c,\tilde{s}^N} \\ & + \beta(1+r) \sum_{\tilde{s}^N\in\mathcal{S}^N} (\Pi^u_{s^N\tilde{s}^N} - \Pi_{s^N\tilde{s}^N})\xi^u_{\tilde{s}^N} U_{c,\tilde{s}^N}.\end{aligned} \tag{54}$$

The second term has just been proved to converge to 0. All previous equations can be stacked such that (54) is written in matrix form. Indeed, a history s^N can be seen as a vector $\{(y_{-N+1}, e_{-N+1}), \ldots, (y_0, e_0)\}$, where $e_k = 0$ if the agent is unemployed and $e_k = 1$

if she is employed, and $y_k = 1, \ldots, Y$ is her productivity level. The number of histories is $N_{tot} = S^N$ ($S = Card(\mathcal{S})$). We can identify each history by the integer $k_{s^N} = 1, \ldots, N_{tot}$:

$$k_{s^N} = \sum_{k=0}^{N-1} N_{tot}^{-N+1-k} \left(e_k \times Y + y_k - 1\right) + 1, \tag{55}$$

which corresponds to an enumeration in base S. In this enumeration, the first $N_{tot}/2$ histories are histories where agents are currently unemployed.

Let $\mathbf{U_c}$, $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^u$ be the N_{tot}-vectors of end-of-period marginal utilities, and ξs coefficients. Define as $\mathbf{I}$ the identity matrix and $\mathbf{\Pi} = (\mathbf{\Pi}_{kk'})_{k,k'=1,\ldots,N_{tot}}$ as the transition matrix from history k to history k'. Equation 54 can be written as:

$$(\mathbf{I} - \beta(1+r)\mathbf{\Pi})((\boldsymbol{\xi}_k^u - \boldsymbol{\xi}_k)\mathbf{U}_{c,k})_k = \beta(1+r)(\sum_{k'=1}^{N_{tot}} (\mathbf{\Pi}_{k,k'}^u - \mathbf{\Pi}_{kk'})\boldsymbol{\xi}_{k'}^u \mathbf{U}_{c,k'})_k,$$

Since $\beta(1+r)$ is independent of N and verifies $\beta(1+r) < 1$, the matrix $\mathbf{I} - \beta(1+r)\mathbf{\Pi}$ is invertible and its eigenvalues are bounded away from 0 for all N. Finally observing that $((\xi_{s^N}^u - \xi_{s^N})U_{c,s^N})_{s^N}$ can be made arbitrarily small and that $\xi_{s^N} \to 1$ concludes the proof.

B.4 Proof of Proposition 3

We now turn to the convergence of allocations computed using a first-order perturbation method. The proof relies on Villemot (2011), who provides a detailed account of the first-order perturbation technique. Using Villemot (2011)'s notation, it is shown that the first-order perturbation method involves writing the vector of endogenous variables, denoted by y_t, as a function of the subset of endogenous variables appearing with a lag (y_{t-1}^-), as well as the vector of exogenous variables u_t. More precisely, at the first-order, the perturbation is defined by the following recursion: $y_t = \overline{y} + g_y y_{t-1}^- + g_u(u_t - \overline{u})$, where quantities with an overbar denotes steady-state values. In our case, all of these quantities apart from u_t (which represents aggregate shocks) depend on the truncation N length, such that the recursion characterizing the perturbated system can be rewritten as $y_t^N = \overline{y}^N + g_y^N y_{t-1}^{N,-} + g_u^N(u_t - \overline{u})$. Nonexplosive dynamics impose the condition that the module of eigenvalues of g_y are smaller than one. We now show that the sequence $(y^N)_N$ can be seen as a Cauchy sequence in $\ell_\infty(\mathbb{N})$ – thereby proving convergence. The key point is to use the first point of the proof of Section B.3.1, stating that any truncation of length $N+1$ is a refinement of a truncation of length N. It is therefore meaningful to consider (up to this transformation) the difference between y_t^N and y_t^{N+1}, which can be written as:

$$y_t^{N+1} - y_t^N = \overline{y}^{N+1} - \overline{y}^N + (g_y^{N+1} - g_y^N)y_{t-1}^{N+1,-} + g_y^N(y_{t-1}^{N+1,-} - y_{t-1}^{N,-}) + (g_u^{N+1} - g_u^N)(u_t - \overline{u}),$$

where the terms $\overline{y}^{N+1} - \overline{y}^N$, $g_y^{N+1} - g_y^N$ and $g_u^{N+1} - g_u^N$ can be made arbitrarily small thanks to the fact that they only depend on steady-state values and that we have a convergence result for steady-state allocations. Since the dynamic system is non-explosive and since the limit of the initial points is well-defined by construction ($\lim_N y_0^{N,-} = y_0^-$), we deduce that $(y_t^N)_N$ is a Cauchy sequence and thereby converges for any t when N increases.

Finally, if the perturbation is well-defined for the full-fledged model (for Blanchard and Kahn (1980) conditions), then it will also be in the truncated model for sufficiently large N.

B.5 Proof of Proposition 4

Only the proof for the steady state needs to be demonstrated. The proof for the perturbation method is not specific and follows the same lines as the proof in Section B.4.

The proof for the steady state runs as follows. Our optimization procedure selects the UI policy, such that the associated truncated Bewley allocations solves the FOCs (17) and (18) – where the ξs are computed from the same Bewley equilibrium allocations. More precisely, the procedure is as follows (for a given N):

1. Select a UI policy (τ, ϕ) such that the UI budget is balanced and (14) holds.
2. Compute the associated Bewley equilibrium and deduce from these allocations, the steady-state truncated allocations $(a_{s^N}, c_{s^N}, l_{s^N})_{s^N \in \mathcal{S}^N}$, as well as the $(\xi_{s^N})_{s^N \in \mathcal{S}^N}$, using the method of Section C.2.1.
3. Use the Ramsey FOC (17) at the steady state to compute the Ramsey Lagrange multipliers $(\lambda_{s^N})_{s^N \in \mathcal{S}^N}$. See Section C.2.2 for further details.
4. If the Ramsey FOC (18) holds, then the steady-state Ramsey equilibrium corresponds to the UI policy under consideration and to the associated Bewley equilibrium. If not, iterate on the UI policy and go back to Step 1.

This enables to characterize the optimal policy (ϕ_N, τ_N) that we index in this section by N.

Points 1 to 3 allow us to write allocations and Lagrange multipliers as a function of the replacement rate ϕ (for any N). In Point 4, the truncated FOC (18) at the steady state implies that for any N, there exists a function $f_{\phi,N}$ such that the optimal replacement rate τ_N is characterized by $f_{\phi,N}(\phi_N) = 0$.

We now focus on the full-fledged Ramsey program of Section 3. We already know that for large N, truncated allocations $(a_{s^N}, c_{s^N}, l_{s^N})_{s^N \in \mathcal{S}^N}$ converge to (exact) Bewley allocations and that $(\xi_{s^N})_{s^N \in \mathcal{S}^N}$ converge to 1. Therefore, (17) is the limit of the truncated FOC (17) at the steady state, which implies that $(\lambda_{t,s^N})_{s^N \in \mathcal{S}^N}$ converge at the steady state to individual Lagrange multipliers. This holds for any replacement rate.

Equation (18) implies that the optimal replacement rate ϕ_* is characterized by $f_\phi(\phi_*) = 0$ (assuming existence of the full-fledged Ramsey equilibrium), where f_ϕ is the limit of $(f_{\tau,N})$ for large N (since these functions solely depend on steady-state allocations, whose limits are full-fledged Bewley allocations). Assuming that $f_{\tau,N}$ and f_ϕ are continuous, we can deduce that $\phi_* = \lim_N \phi_N$. The continuity of the functions $f_{\tau,N}$ and f_ϕ is a consequence of the Berge theorem – which requires the compactness of the choice set. This is guaranteed by the upper bound $a_{\max}$ on saving choices and the Tychonoff theorem. This concludes the proof and states the truncated Ramsey equilibria at the steady state converge to the (steady-state) full-fledged Ramsey equilibria.

C Details of the implementation of the truncation method

This section is organized in three parts. Section C.1 presents all the equations driving the dynamics of the truncated model. Section C.2 details the computations that enable to express the steady-state value of the ξs using linear algebra. Section C.3 describes a detailed version of the algorithm for solving the Ramsey program.

C.1 Summary of the dynamics of the truncated model

The system characterizing the dynamics of the model in the presence of an optimal time-varying replacement rate can be written as follows. We start with equations valid for all $s^N \in \mathcal{S}^N$:

$$\begin{aligned}
l_{t,s^N} &= \chi^{\varphi}(1-\tau_t)^{\varphi} w_t^{\varphi} y_0^{N,\varphi} 1_{e_0^N=e}, \\
c_{t,s^N} + a_{t,s^N} &\le (1+r_t)\tilde{a}_{t,s^N} + \Big((1-\tau_t)1_{e_0^N=e} + \phi_t 1_{e_0^N=u}\Big) l_{t,s^N} y_0^N w_t, \\
\tilde{a}_{t,s^N} &= \sum_{\tilde{s}^N\in\mathcal{S}^N} \Pi_{t,s^N\tilde{s}^N} \frac{S_{\tilde{s}^N}}{S_{s^N}} a_{t-1,\tilde{s}^N}, \\
\Psi_{t,s^N} &= U_{c,t,s^N} - \Big(\lambda_{t,s^N} - (1+r_t)\tilde{\lambda}_{t,s^N}\Big) \xi_{s^N} U_{cc,t,s^N},
\end{aligned}$$

Then, the equations valid for unconstrained histories only ($s^N \notin \mathcal{C}$) are:

$$\begin{aligned}
&\xi_{s^N} U_{c,t,s^N} = \beta(1+r) \sum_{\tilde{s}^N\in\mathcal{S}^N} \Pi_{t+1,s^N\tilde{s}^N} \xi_{\tilde{s}^N} U_{c,t+1,\tilde{s}^N} \\
&\Psi_{t,s^N} = \beta \sum_{\tilde{s}^N\in\mathcal{S}^N,} \Pi_{t+1,s^N\tilde{s}^N} \mathbb{E}_t\Big[(1+r_{t+1})\Psi_{t+1,\tilde{s}^N}\Big] + \beta\frac{1-\alpha}{\varphi}\mathbb{E}_t\Bigg[\frac{1}{L_{t+1}}\left(\frac{K_t}{L_{t+1}}\right)^{\alpha-1} \\
&\times \sum_{\tilde{s}^N\in\mathcal{S}^N} \Bigg((1-\tau_{t+1})1_{\tilde{e}_0^N=e} + \frac{S_{t+1,e}}{S_{t+1,u}}(\tau_{t+1}(1+\varphi)-1)1_{\tilde{e}_0^N=u}\Bigg)\Psi_{t+1,\tilde{s}^N} S_{t+1,\tilde{s}^N} l_{t+1,\tilde{s}^N} \tilde{y}_0^N\Bigg],
\end{aligned}$$

For constrained histories ($s^N \in \mathcal{C}$), we have $a_{t,s^N} + \bar{a} = \lambda_{t,s^N} = 0$. Aggregate equations are:

$$\sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \tilde{\lambda}_{t,s^N} \xi_{s^N} U_{c,t,s^N} = - \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} \Psi_{t,s^N} \tilde{a}_{t,s^N} + \frac{1}{\alpha\varphi} \frac{K_{t-1}}{L_t} \tag{56}$$

$$\times \sum_{s^N \in \mathcal{S}^N} \left(-(1-\tau_t) 1_{e_0^N = e} + \frac{S_{t,e}}{S_{t,u}} \left(1 + \alpha\varphi - (1+\varphi)\tau_t\right) 1_{e_0^N = u} \right) S_{t,s^N} \Psi_{t,s^N} l_{t,s^N} \tilde{y}_0^N,$$

$$K_t = \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} a_{t,s^N},\ L_t = \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} y_0^N l_{t,s^N},\ \text{and} \sum_{s^N \in \mathcal{S}^N} S_{t,s^N} c_{t,s^N} + K_t = Y_t + K_{t-1},$$

$$r_t = \alpha Z_t \left(\frac{K_{t-1}}{L_t} \right)^{\alpha - 1} - \delta \text{ and } w_t = (1-\alpha) Z_t \left(\frac{K_{t-1}}{L_t} \right)^{\alpha}.$$

The optimal replacement rate is given by equation (56), corresponding to the first-order condition of the Ramsey program of Section 4.4. Note that the dynamics of the model with an exogenous replacement rate can be deduced from the previous set of equations, in which ϕ (and τ) has to be set to its exogenous value and equation (56) is discarded.

C.2 Matrix representation of the steady state to compute ξ and λ

Before turning to the matrix representation, we introduce the following notation: $\circ$ is the Hadamard product, the Kronecker product, and $\times$ the usual matrix product. For a vector V, $diag(V)$ is the diagonal matrix with V on the diagonal.

Matrix notations have been introduced in Appendix B.5. Recall that each history is identified by the integer $k_{s^N} = 1, \ldots, N_{tot}$ defined in (55), with $N_{tot} = S^N$.

C.2.1 Computing the ξs

Let $\mathbf{S}$ be the N_{tot}-vector of steady-state history sizes. Similarly, let $\mathbf{a}$, $\mathbf{c}$, $\boldsymbol{\ell}$, $\boldsymbol{\nu}$, $\mathbf{U_c}$, $\mathbf{U_{cc}}$ be the N_{tot}-vectors of end-of-period wealth, consumption, labor supply, Lagrange multipliers, marginal utilities, and derivatives of the marginal utility, respectively. Each vector is computed using equation (22) as the aggregation of the relevant variable – known from the steady-state Bewley mode. We also define:

$$\mathbf{W} = w \begin{bmatrix} \phi \\ 1-\tau \end{bmatrix} \otimes \boldsymbol{y} \otimes 1_B,\ \mathbf{L}_e = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \boldsymbol{y} \otimes 1_B,\ \mathbf{L}_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \boldsymbol{y} \otimes 1_B,$$

where $\boldsymbol{y} = [y_1 \ldots y_Y]^\top$ is the vector of productivity levels. Let $\mathbb{P}$ be the diagonal matrix having 1 on the diagonal at s^N if and only if the history s^N is not credit constrained (i.e., $\nu_{s^N} = 0$), and 0 otherwise. Similarly, define $\mathbb{P}^c = \mathbf{I} - \mathbb{P}$, where $\mathbf{I}$ is the $(N_{tot} \times N_{tot})$-identity matrix. Let Π be the transition matrix across histories. In the steady state:

$$\mathbf{S} = \Pi\mathbf{S} \text{ and } \mathbb{P}^c\mathbf{a} = -\bar{a}1_{N_{tot}\times 1}, \tag{57}$$
$$\mathbf{S}\circ\mathbf{c} + \mathbf{S}\circ\mathbf{a} = (1+r)\Pi\left(\mathbf{S}\circ\mathbf{a}\right) + \left(\mathbf{S}\circ\mathbf{W}\circ\boldsymbol{\ell}\right),$$
$$\left(\frac{r+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}}\mathbf{L_e}^\top\times\mathbf{S} = \mathbf{S}^\top\times\mathbf{a},$$
$$\tau = \phi\frac{\mathbf{L_u}^\top\times\mathbf{S}}{\mathbf{L_e}^\top\times\mathbf{S}}, \text{ and } w = (1-\alpha)\left(\frac{r+\delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}.$$

(The previous equalities can be double-checked numerically). We define the vector $\tilde{\boldsymbol{\xi}}$ as:

$$\tilde{\boldsymbol{\xi}} = \left[\mathbb{P}\left(diag\left(u'\left(\mathbf{c}\right)\right) - \beta(1+r)\Pi\times diag\left(u'\left(\mathbf{c}\right)\right)\right) + \mathbb{P}^c\right]^{-1}\boldsymbol{\nu}, \tag{58}$$

which is well-defined since the matrix $\mathbb{P}\left(diag\left(u'\left(\mathbf{c}\right)\right) - \beta(1+r)\Pi\times diag\left(u'\left(\mathbf{c}\right)\right)\right) + \mathbb{P}^c$ is invertible (because $\beta\left(1+r\right) < 1$, and $\mathbb{P}+\mathbb{P}^c = \mathbf{I}$), and since the vector $\boldsymbol{\nu}$ is not zero (because some histories are credit constrained due to a credit limit above the natural borrowing limit). With definition (58), we can check that for unconstrained histories, we have:

$$\tilde{\boldsymbol{\xi}}\circ u'\left(\mathbf{c}\right) = \beta(1+r)\Pi\left(\tilde{\boldsymbol{\xi}}\circ u'\left(\mathbf{c}\right)\right),$$

and for constrained histories, $\tilde{\boldsymbol{\xi}}\circ u'\left(\mathbf{c}\right) = \beta(1+r)\Pi\left(\tilde{\boldsymbol{\xi}}\circ u'\left(\mathbf{c}\right)\right) + \boldsymbol{\nu}$, where we use $\tilde{\boldsymbol{\xi}}\circ u'\left(\mathbf{c}\right) = diag\left(u'\left(\mathbf{c}\right)\right)\tilde{\boldsymbol{\xi}}$. These two properties being invariant after a positive rescaling, we finally define $\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}}/sum\left(\mathbf{S}\circ\tilde{\boldsymbol{\xi}}\right)$ with $sum\left(\mathbf{S}\circ\boldsymbol{\xi}\right) = 1$ ($sum(\boldsymbol{x}) = \sum_{x\in\boldsymbol{x}} x$ for any vector $\boldsymbol{x}$).

C.2.2 Computing equilibrium Lagrange multipliers λ

We derive here the Lagrange multipliers of the Ramsey program as a function of the steady-state solution (i.e., allocations and prices), which is assumed to be known. Denoting the vectors associated with the Lagrange multipliers by $\boldsymbol{\lambda}$, $\tilde{\boldsymbol{\lambda}}$, and $\boldsymbol{\Psi}$, we have: $\boldsymbol{\Psi} = \boldsymbol{\xi}\circ\mathbf{U_c} - \left(\boldsymbol{\lambda} - (1+r)\tilde{\boldsymbol{\lambda}}\right)\circ\boldsymbol{\xi}\circ\mathbf{U_{cc}}$, from (16). We define the matrix Π^Λ by $\Pi^\Lambda_{\tilde{k}k} = \frac{S_k\Pi_{k\tilde{k}}}{S_{\tilde{k}}}$, such that $\tilde{\boldsymbol{\lambda}} = \Pi^\Lambda\boldsymbol{\lambda}$, and the matrix Π^Ψ by:

$$\Pi^\Psi_{k\tilde{k}} = \beta(1+F_K)\Pi_{k\tilde{k}} + \beta\frac{1-\alpha}{\alpha\varphi}\frac{r+\delta}{L}\left((1-\tau)1_{\tilde{k}>\frac{N_{tot}}{2}} + \frac{S_e}{S_u}\left(\tau\left(1+\varphi\right)-1\right)1_{\tilde{k}\leq\frac{N_{tot}}{2}}\right)S_{\tilde{k}}l_{\tilde{k}}y_{\tilde{k}},$$

where $1_{\tilde{k}>\frac{N_{tot}}{2}}$ represents employed agents and $1_{\tilde{k}\leq\frac{N_{tot}}{2}}$ unemployed ones. It can be checked from (17) that $\mathbb{P}\boldsymbol{\Psi} = \mathbb{P}\Pi^\Psi\boldsymbol{\Psi}$ and that the vector of the Lagrange multipliers, $\boldsymbol{\lambda}$, verify:

$$\boldsymbol{\lambda} = \left[\mathbb{P}^c + \mathbb{P}(\mathbf{I}-\Pi^\Psi)\left(diag\left(\boldsymbol{\xi}\circ\mathbf{U_{cc}}\right)\left(\mathbf{I}-(1+F_K)\Pi^\Lambda\right)\right)\right]^{-1}\mathbb{P}(\mathbf{I}-\Pi^\Psi)(\boldsymbol{\xi}\circ\mathbf{U_c}). \tag{59}$$

Importantly, the right-hand side can be deduced from the Bewley allocations, which makes the computation of $\boldsymbol{\lambda}$ straightforward. We then deduce $\left(\tilde{\boldsymbol{\lambda}}, \boldsymbol{\Psi}\right)$ with:

$$\tilde{\boldsymbol{\lambda}} = \Pi^\Lambda\lambda, \text{ and } \boldsymbol{\Psi} = \xi\circ\mathbf{U}_c - \xi\circ\mathbf{U_{cc}}\circ\left(\mathbf{I}-(1+F_K)\Pi^\Lambda\right)\lambda. \tag{60}$$

Equation (33) that allows to check for the optimality of the planner's instruments is:

$$\mathbf{V}^\top \times 1_{N_{tot}\times 1} = 0, \tag{61}$$

$$\text{where: } \mathbf{V} \equiv \frac{\varphi\alpha}{K}(\mathbf{S}\circ\Psi\circ\tilde{\mathbf{a}} + \mathbf{S}\circ\Lambda\circ\xi\circ\mathbf{U}_c) + \frac{1-\tau}{L}\mathbf{S}\circ\Psi\circ\mathbf{L}^e - (1-\tau+\varphi(\alpha-\tau))\frac{S_e}{S_u}\mathbf{S}\circ\Psi\circ\mathbf{L}^u.$$

C.3 Algorithm for solving the Ramsey problem

The algorithm for computing Ramsey policies is as follows.

1. Choose a length of the truncation N.
2. Set a reasonable initial value for the replacement rate ϕ (we start with $\phi = 50\%$).
   (a) Solve the general Bewley model for the given value of ϕ.
   (b) Use the steady-state outcome to compute the truncated allocation: $\mathbf{a}$, $\mathbf{c}$, $\boldsymbol{\ell}$, $\boldsymbol{\nu}$, $\mathbf{U_c}$, $\mathbf{U_{cc}}$. Then deduce $\tilde{\boldsymbol{\xi}}$ and $\boldsymbol{\xi}$ from equations (58).
   (c) Determine the steady-state values of the multipliers $\boldsymbol{\lambda}, \tilde{\boldsymbol{\lambda}}$, and $\boldsymbol{\Psi}$ from Section C.2.2.
3. Iterate on ϕ and repeat Step 2 until equality (61) holds (with some precision criterion).

Once the steady state and the partition have been determined, it is easy to simulate the model using standard perturbation techniques with software such as Dynare (see Adjemian et al., 2011). Simulating the whole optimal allocation for the calibrated economy with aggregate shocks takes less than 1 minute on a standard laptop.

D Supplemental numerical results

D.1 Computational methods

For all simulation methods, we first solve for the steady-state allocation of the full-fledged Bewley model. The consumer's problem is solved on a grid using the Endogenous Grid Point Method (EGM) of Carrol (2006). See also Den Haan (2010) for a presentation. For the decision rules, the asset grid has 50 points, non-linearly spaced, as in Boppart et al. (2018), and households can choose points off the grid by linear interpolation. There are 7 different productivity levels and 2 employment status. We thus solve for 14 policy rules. The Euler equation errors reported in Section 5.5 show that the accuracy is satisfactory.

Truncated model. To derive the truncated model for a given N, we first use the steady-state distribution of the full-fledged Bewley model to compute average consumption and savings in each of the 14 idiosyncratic states. We then use the policy rules $g_i(a)$ for savings and $g_i^c(a)$ for consumption, together with the transition probabilities, $(\Pi_{i,j})_{i,j=1,\ldots,14}$, to compute the average consumption and saving levels for each idiosyncratic history. For example, when we know the steady-state beginning-of-period distribution of wealth, $\Lambda_i(a)$, of agents in states $i = 1, \ldots, 14$, we can compute the steady-state distribution of wealth of agents with history (i, j), $\Lambda_{(i,j)}(a)$ by computing:

$$\Lambda_{(i,j)}(a') = \Pi_{i,j} \int_{a, g_i(a)=a'} \Lambda_i(a)\, da, \tag{62}$$

Once we have the distributions $\Lambda_{(i,j)}$, the average consumption levels and savings by history are simply constructed by $c_{i,j} = \int_a g_j^c(a)\, \Lambda_{(i,j)}(a)\, da$. and $a_{i,j} = \int_a a\Lambda_{(i,j)}(a)\, da$ (note that we use the policy rule of the last states). For histories (i, j, k) with $i, j, k = 1, \ldots, 14$, we start from $\Lambda_{(i,j)}$ and construct $\Lambda_{(i,j,k)}$ as in (62). This strategy allows us to construct recursively the steady-state distribution of wealth for any idiosyncratic history of arbitrary length. Once the average values are computed, we deduce the ξs using equation (58). Note that the Dynare solver can be used to double-check that the steady state computed in Section C.2 is indeed a steady state of the dynamic equations. Finally, we then compute a first-order approximation for the aggregate shock of the whole system of equations, given in Section C.1, using Dynare again.

Reiter model. We implement the algorithm described in Reiter (2009), which is now standard. For each asset level and for each idiosyncratic state, we perform a first-order approximation of the policy rule for the aggregate states. We use these approximated policy rules to simulate the dynamics of the model for 10,000 periods.

Boppart, Krusell, and Mitman. Following Boppart et al. (2018), we first simulate an unexpected shock (MIT shock) to the innovation ε_t^z, to compute the IRFs for the various variables of interest. The transition path is then solved by iteration on the capital path, assuming that the economy comes back to its steady state after 400 periods. These IRFs are then used as numerical partial derivatives for any variable x_t under consideration according to the aggregate shock (which is continuous), at different time-horizons, namely $\frac{\partial x_{t+k}}{\partial \varepsilon_t^z}$ for $k = 0, \ldots, T$ (where T is chosen high enough for the derivative to be negligible for $k > T$). These derivatives are then used to simulate the economy with aggregate shocks, using a Taylor-expansion: $x_t = \bar{x}^{ss} + \sum_{k=0}^{T} \frac{\partial x_t}{\partial \varepsilon_{t-k}^z} \varepsilon_{t-k}^z$ for the simulated history of the innovation ε_t^z.

D.2 Comparison with other solution methods

We compare here three different computational solutions: the Reiter method, the truncated method, and the method of Boppart et al. (2018) – described in Section D.1.

We simulate the same economy as in Section 5.3, with the same parameters. However, we focus on TFP shocks only ($\sigma_{ue} = 0$), as in Boppart et al. (2018) to simplify the comparison. The results are reported in Table 7 (Simulations 1 to 4) The first column describes the computed statistics (using simulations with 10,000 periods). Subsequent columns correspond (in this order) to the Boppart et al. (2018) methodology, the Reiter method, the truncation method, and the representative-agent (RA) economy. The three methods (Reiter, BKM, truncation) yield very similar results – and are very different from the RA economy. For instance the normalized standard deviation of GDP is between 1.75% and 1.79% for the first three methods, whereas it is 2.10% in the RA case.

Methods		BKM	Reiter	Trunc.	RA	BKM	Reiter	Trunc.
Simulations		(1)	(2)	(3)	(4)	(5)	(6)	(7)
Y	mean	1.17	1.17	1.17	1.08	1.11	1.11	1.11
	std/mean (%)	1.75	1.77	1.79	2.10	1.78	1.78	1.77
C	mean	0.84	0.84	0.84	0.80	0.82	0.82	0.82
	std/mean (%)	1.45	1.45	1.45	1.73	1.47	1.48	1.48
L	mean	0.30	0.30	0.30	0.30	0.29	0.29	0.29
	std/mean (%)	0.58	0.59	0.60	.70	0.59	0.59	0.59
K	mean	13.14	13.14	13.14	11.09	11.77	11.77	11.77
	std/mean (%)	1.61	1.62	1.69	2.18	1.64	1.66	1.61
$corr(C, C_{-1})$	(in %)	99.13	99.13	99.17	99.66	99.15	99.15	99.12
$corr(Y, Y_{-1})$		97.48	97.49	97.55	98.19	97.52	97.53	97.50
$corr(C, Y)$		96.17	96.21	96.07	94.59	96.28	96.25	96.27

Table 7: Moments of the simulated model for different resolution techniques.

Comparison for a lower persistence of idiosyncratic shocks. In the benchmark economy, the persistence of the idiosyncratic risk is high, consistently with the data (see Table 1). One may want to check that the accuracy of the truncation method does not depend on this high persistence. To do so, we simulate the same economy with the same parameters, except that we set the persistence of the idiosyncratic risk to $\rho_y = 0.66$ (instead of $\rho_y = 0.992$). Results are reported in simulations (5)–(6) of Table 7. Once again, it can be checked that the three solution methods generate very similar results.

D.3 Additional numerical checks: Alternative rules

We investigate the optimality of the replacement rule (34). We implement two variations of the rule and quantify their impact on aggregate welfare, which is computed using the Reiter method. We simulate all economies for 10,000 periods and then compute relevant moments and welfare. The first variation is a procyclical replacement rate (with similar variances). The coefficients of the rule (34) are $(\phi^{ss}, a_1^{\phi}, a_2^{\phi}, a_0^{\varepsilon}, a_1^{\varepsilon}, a_2^{\varepsilon}, a^K) = (0.4200, -24.4772, 0.0104, -0.6048, 0.3317, -0.3669, 0.0326)$. The results are reported in column (2) of Table 8. Second, we implement the rule with higher variances. The coefficients are $(\phi^{ss}, a_1^{\phi}, a_2^{\phi}, a_0^{\varepsilon}, a_1^{\varepsilon}, a_2^{\varepsilon}, a^K) = (0.4200, -24.4772, 0.0104, 0.9048, -0.4517, 0.5469, -0.04726)$. The results are reported in column (3). For each rule, we report the same statistics as in Table 4, as well as the welfare difference with the benchmark rule (column (1) and equation (34)). The welfare is computed in period 0 using an intertemporal utilitarian welfare criterion. We report the increase in consumption Δ^c of all agents in all periods (such that the agents have a period utility $u((1+\Delta^c)c_t^i - \chi^{-1}\frac{(l_t^i)^{1+1/\varphi}}{1+1/\varphi}))$ for the total intertemporal

Simulations		(1)	(2)	(3)
		Bench.	Procyclical	High variance
Y	mean	1.17	1.17	1.17
	std/mean (%)	2.71	1.49	2.99
C	mean	0.84	0.84	0.84
	std/mean (%)	2.27	1.20	2.52
L	mean	0.30	0.30	0.30
	std/mean (%)	1.70	0.31	2.02
K	mean	13.14	13.14	13.14
	std/mean (%)	2.45	2.27	2.67
ϕ	mean(ϕ)(%)	42	42	42
	std(ϕ)(%)	23	25	35
$corr(C, C_{-1})$	(%)	99.18	98.50	99.17
$corr(Y, Y_{-1})$	(%)	97.52	97.28	97.47
$corr(C, Y)$	(%)	95.30	98.13	94.87
$corr(Y, \Phi)$	(%)	-96.65	96.87	-0.9628
Equ. cons. increase Δ^c	(%)	-	2.0	1.7

Table 8: Impact of different rules for the replacement rate.

welfare to be the same in economies (2) and (3) as in the benchmark economy (1). Both rules imply a decrease in welfare compared to the benchmark economy, by around 1% or 2% of consumption equivalent, thereby confirming the optimality of the replacement rate implied by the Ramsey program.

E Supplemental theoretical results

This section contains three parts. Section E.1 provides a simple example to illustrate why the sign of the Lagrange multiplier on the Euler equation of agent i can be interpreted as the perception by the planner of the quantity of savings of agent i. Section E.2 presents an economy in which credit constraints have been substituted by penalty functions. Finally, Section E.3 generalizes the truncation theory to non-GHH utility functions.

E.1 Understanding Lagrange multipliers on Euler equations

The analysis of the main text uses Lagrange multipliers on Euler equations and claims that these multipliers can be either positive or negative and that their sign is related to the distortions on the saving incentives (from the planner's point of view). This section provides a very simple example (textbook style) to illustrate this statement. In addition, it clarifies some properties of exterior penalty functions, which are used in Section E.2 below.

Consider an economy where the planner has an instrument τ, and the agent can choose a variable a. The agent maximizes a concave objective with a constraint $a \geq -\bar{a}$:

$$\max_{a\in\mathbb{R}} \; -(a-\tau)^2 - \tau^2, \quad \text{s.t. } a \geq -\bar{a},$$

which yields the FOC (ν being the credit-constraint Lagrange multiplier):

$$a - \tau = \nu. \tag{63}$$

This corresponds to $a = \tau$ if $\tau \geq -\bar{a}$ or $a = -\bar{a}$ if $\tau < -\bar{a}$.

The planner's program can now be written as:

$$\max_{a,\tau\in\mathbb{R}} \; -(a-\tau)^2 - \tau^2$$
$$\text{s.t. } a - \tau = \nu \text{ and } a \geq -\bar{a}.$$

The Lagrangian is (λ is the Lagrange multiplier on the agent's FOC and μ the multiplier on the credit-constraint): $\mathcal{L} = -(a-\tau)^2 - \tau^2 - 2\lambda(a-\tau) - 2\mu(a+\bar{a})$, which yields:

$$a - \tau + \lambda + \mu = 0 \tag{64}$$
$$-a + 2\tau - \lambda = 0 \tag{65}$$

Note that we have $\lambda\mu = 0$ and $\lambda\nu = 0$. There are two possible cases, depending on whether the constraint $a \geq -\bar{a}$ is binding or not (i.e., whether $\lambda \neq 0$ or not):

1. $\lambda = 0$. So, we have $\mu = \nu = 0$. Equation (63) implies $a = \tau$ and equations (64) and (65) become: $\lambda = 0$ and $\tau - \lambda = 0$. So the solution is:

$$\lambda = \nu = \mu = 0, \text{ and } a = \tau = 0. \tag{66}$$

2. $\lambda \neq 0$. So $a = -\bar{a}$ and equations (64) and (65) become:

$$-\mu = \nu = -\frac{\bar{a}}{2}, \text{ and } \tau = -\frac{\bar{a}}{2}. \tag{67}$$

Penalty functions

We define the penalty function g as follows:

$$g(a) = \frac{1}{2}\max(-\bar{a}-a,0)^2, \quad g'(a) = -\max(-\bar{a}-a,0), \quad g''(a) = 1_{a\leq-\bar{a}}.$$

Agent's program. Let γ be the weight of the penalty function, the agent's program is: $\max_{a\in\mathbb{R}} \; -(a-\tau)^2 - \tau^2 - 2\gamma g(a)$. The FOC yields:

$$a - \tau - \gamma\max(-\bar{a}-a,0) = 0 \tag{68}$$

So, there are two solutions: (i) $a = \tau$ if $-\bar{a}-\tau \leq 0$, or $a = \frac{\tau}{1+\gamma} + \frac{\gamma}{1+\gamma}(-\bar{a})$ if $-\bar{a}-\tau > 0$. We can observe than when $\gamma \to \infty$, these two solutions converge to the solutions (66) and (67) of the Ramsey problem.

Planner's program. For the sake of generality, we consider a penalty function with a different coefficient $\tilde{\gamma} = \kappa\gamma$, where we can have $0 < \kappa \leq 1$ or $\kappa > 1$, depending on who (the agent of the planner) gives the constraint the highest value. The planner's program is: $\max_{a,\tau\in\mathbb{R}} -(a-\tau)^2 - \tau^2 - 2\tilde{\gamma}g(a)$, s.t. $a - \tau = -\gamma g'(a)$. The Lagrangian is: $\mathcal{L} = -(a-\tau)^2 - \tau^2 - 2\lambda(a - \tau + \gamma g'(a)) + 2\tilde{\gamma}g(a)$, with the FOCs:

$$0 = \frac{a+\lambda}{2} + \lambda\gamma 1_{a\leq -\bar{a}} - \tilde{\gamma}\max(-\bar{a} - a, 0), \tag{69}$$

$$\tau = \frac{a+\lambda}{2}. \tag{70}$$

Again, there are two cases:

1. $a > -\bar{a}$. Then (68), (69) and (70) imply: $a = \tau = \frac{a+\lambda}{2} = 0$.

2. $a \leq -\bar{a}$. Then (68) implies $\tau = (\gamma+1)a + \gamma\bar{a}$. We obtain using (69) and (70):

$$a = \frac{2\gamma^2 + (\kappa+1)\gamma}{2\gamma^2 + (\kappa+2)\gamma + 1}\bar{a},\ \tau = -\frac{\gamma^2 + \kappa\gamma}{2\gamma^2 + (\kappa+2)\gamma + 1}\bar{a},\ \lambda = -\frac{(\kappa-1)\gamma}{2\gamma^2 + (\kappa+2)\gamma + 1}\bar{a}. \tag{71}$$

Remark 1 (Sign of λ) *The sign of λ can be positive or negative, depending on whether $\kappa > 1$ or $\kappa < 1$, which indicates who (between the planner and the agent) gives the credit constraint the highest value.*

At the limit: $a \to -\bar{a},\ \lambda \to 0,\ \tau \to -\frac{\bar{a}}{2}$, which is the solution of the initial program.

E.2 Penalty functions

In this section, we replace the credit constraint with a penalty function denoted by $g(a)$ for a saving choice a. The functional form of g is standard for exterior penalty function (see Luenberger and Ye, 2016 for a textbook treatment of penalty functions). For a saving $a < -\bar{a}$, the distance to the credit limit is $-\bar{a} - a$. The baseline penalty function is thus $g(a) = \max(-\bar{a} - \underline{a}, 0)^2$. We parametrize this penalty function by a scalar $\gamma > 0$ such that all agents face a penalty function $\gamma g(a)$. We denote with a γ superscript the quantities associated to penalty function $\gamma g(a)$. The goal of this section is to show that the FOCs (38) and (39) are limits of FOCs for infinitely concave penalty functions, i.e., for $\gamma \to \infty$.

E.2.1 The truncated economy

In the presence of the penalty function, the individual program can be written as

$$\max_{(c_t^{\gamma,i}, l_t^{\gamma,i}, a_t^{\gamma,i})_{t\geq 0,i}} \tilde{J}_i = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\xi_{s^N} U(c_t^{\gamma,i}, \hat{l}_t^{\gamma,i}) - \gamma g(a_t^{\gamma,i})\right)\Big], \tag{72}$$

$$a_t^{\gamma,i} + c_t^{\gamma,i} = ((1-\tau_t) l_t^{\gamma,i} 1_{e_0^N = e} + \phi_t l_{t,s^N,e}^{\gamma} 1_{e_0^N = u}) y_0^N w_t^{\gamma} + (1+r_t) a_{t-1}^{\gamma,i}, \tag{73}$$

$$c_t^{\gamma,i},\ l_t^{\gamma,i} \geq 0 \tag{74}$$

with given initial conditions. Compared to the initial program (5)–(7), the objective function (72) includes penalty functions, while credit constraint have been removed.

As the replacement rate is exogenous, we make the following additional assumption.

Assumption B *We assume that for all $t \geq 0$: $1-\tau_t > \frac{\varphi}{1+\varphi}$ and $\phi_t > \frac{\varphi}{1+\varphi}$.*

Assumption B is purely technical: it guarantees that the utility – with a GHH utility function – of employed and unemployed agents in autarky is well-defined and finite. This assumption is obviously verified in our numerical exercise of Section 5.

Lemma 1 *For any $\gamma > 0$, we have $\tilde{J}_\gamma > -\infty$, where $\tilde{J}_\gamma$ is defined in equation (72).*

Lemma 1 states that the individual welfare of the economy is well-defined and finite for any values of γ. It is a direct consequence of Assumption B.

Proof. We prove that the autarky allocation (i.e., null savings at all dates) is feasible. We assume that $a_t^{aut,i} = 0$, for all $t \geq 0$ and all i. The consumption of an agent when employed and with productivity y is $c_t^{aut,i} = (1-\tau) l_t^{aut,i}(y) y w_t$. Since her labor supply is $l_t^{aut,i}(y) = (\chi w_t y)^\varphi$, we have:

$$c_t^{aut,i} - \frac{1}{\chi}\frac{l_t^{aut,i}(y)^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}} = \left(1-\tau-\frac{\varphi}{1+\varphi}\right)\frac{(\chi w_t y)^{1+\varphi}}{\chi}, \tag{75}$$

Similarly for the unemployment case, from which $\zeta(y) = l(y)$:

$$c_t^{aut,i} - \frac{1}{\chi}\frac{l_t^{aut,i}(y)^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}} = \left(\phi-\frac{\varphi}{1+\varphi}\right)\frac{(\chi w_t y)^{1+\varphi}}{\chi}, \tag{76}$$

Assumption B ensures that quantities (75) and (76) are bounded away from zero for all γ. The autarky allocation is thus feasible (independently of γ), and has a finite welfare, which concludes the proof. ■

We now derive the FOCs in the truncated economy and investigate their convergence when $\gamma \to \infty$. Following the same steps as in Secion 4, we deduce from the program (72) the following Euler equations:

$$l_{t,s^N}^\gamma = \chi(1-\tau_t) y_0^N w_t U_{c,t,s^N}^\gamma, \tag{77}$$

$$\xi_{s^N} U_{c,t,s^N}^\gamma = \gamma g'(a_{t,s^N}^\gamma) + \beta \mathbb{E}_t\left[\sum_{\tilde{s}^N} \Pi_{t+1,s^N,\tilde{s}^N}(1+r_{t+1})\xi_{\tilde{s}^N} U_{c,t+1,s^N}^\gamma\right], \tag{78}$$

with $U_{c,t,s^N}^\gamma = U_{c,t,s^N,\gamma}(c_{t,s^N}^\gamma, l_{t,s^N}^\gamma)$. Compared to Euler equations (38) and (39), the labor Euler equation (77) remains unchanged, while the consumption Euler equation (78) does not feature a Lagrange multiplier for the credit constraint but instead the derivative of the penalty function. All other equations characterizing the truncated equilibrium are unchanged (factor prices, market clearing conditions, etc.).

We can now state our result regarding the limit Euler equation.

Lemma 2 (Limit penalty) *When $\gamma \to \infty$, the solution of (35) is such that:* $\lim_{\gamma\to\infty} a^{\gamma}_{t,s^N} = -\bar{a}$ *or, defining* $c^{\infty}_{t,s^N} = \lim_{\gamma\to\infty} c^{\gamma}_{t,s^N}$ *and* $\hat{l}^{\infty}_{t,s^N} = \lim_{\gamma\to\infty} \hat{l}^{\gamma}_{t,s^N}$*:*

$$\xi_{s^N} U_c(c^{\infty}_{t,s^N}, \hat{l}^{\infty}_{t,s^N}) = \beta \mathbb{E}_t\Big[\sum_{\tilde{s}^N} \Pi_{t+1,s^N,\tilde{s}^N}(1+r_{t+1})\xi_{\tilde{s}^N} U_c(c^{\infty}_{t+1,s^N}, \hat{l}^{\infty}_{t+1,s^N})\Big]. \tag{79}$$

Lemma 2 states that, when the penalty function becomes infinitely concave, then either the borrowing of agents facing a positive penalty tends toward the borrowing limit, or their limit allocation verifies Euler equation (79), which is the same as Euler equation (8) in the baseline truncated economy.

Proof. The proof is performed by contradiction. Assume that there exists $s^N \in \mathcal{S}^N$, such that $\lim_{\gamma\to\infty} a_{t,s^N} < -\bar{a}$. Then, there exists $\varepsilon > 0$, such that $a_{t,s^N} \leq -\bar{a} - \varepsilon$, for γ high enough, which implies that $\lim_{\gamma\to\infty} \gamma g(a_{t,s^N}) = \lim_{\gamma\to\infty} \gamma\varepsilon^2 = \infty$. Hence, $\lim_{\gamma\to\infty} \tilde{J}_{\gamma} = -\infty$, which contradicts Lemma 1. The second part stems from (78), as $\gamma g'(a^{\gamma}_{t,s^N}) = 0$ if $a^{\gamma}_{t,s^N} > -\bar{a}$. ■

E.2.2 The Ramsey program

We now rewrite the Ramsey program in the presence of penalty functions. The planner's program can be written as follows – we drop the dependence in γ to lighten the notation:[22]

$$\max_{\left((a_{t,s^N}, c_{t,s^N}, l_{t,s^N})_{s^N}, \phi_t, \tau_t\right)_{t\geq 0}} \mathbb{E}_0\Big[\sum_{t=0}^{\infty} \beta^t \sum_{s^N\in\mathcal{S}^N} S_{t,s^N}\left(\xi_{s^N} U(c_{t,s^N}, \hat{l}_{t,s^N}) - \gamma g(a_{t,s^N})\right)\Big], \tag{80}$$

$$\text{s.t. } \xi_{s^N} U_{c,t,s^N} = \gamma g'(a_{t,s^N}) + \beta \mathbb{E}_t\Big[(1+r_{t+1}) \sum_{\tilde{s}^N\in\mathcal{S}^N} \Pi_{t+1,s^N\tilde{s}^N} \xi_{\tilde{s}^N} U_{c,t+1,\tilde{s}^N}\Big], \tag{81}$$

and subject to the same equations as in the main text: (24), (28), (14), and (3). There are only two differences compared with the Ramsey program in the main text: (i) the presence of penalty functions in the planner's objective; and (ii) penalty functions in the Euler equations (81). We can now state our main equivalence result.

Proposition 6 (Equivalence result) *The solution of program (29) is a solution of the program (80) when the penalty function become infinitely concave.*

E.2.3 Proof of Proposition 6

Using Lemma 2, it only remains to be proven that the FOCs of the Ramsey program (80) converge to those of the Ramsey program (29) when penalty costs become infinitely large.

[22] As illustrated in Section E.1, we can choose the same penalty γ for the agents and the planner.

Rewriting the Ramsey program. Denoting by $\beta^t S_{t,s^N}\lambda_{t,s^N}$ the Lagrange multiplier on (81) and using (31), the planner's objective, denoted by J_γ, can be expressed as:

$$J_\gamma = \mathbb{E}_0 \sum_{t,s^N} \beta^t S_{t,s^N}\xi_{s^N}\left(U_{t,s^N} - \left(\lambda_{t,s^N} - (1+r_t)\tilde{\lambda}_{t,s^N}\right)U_{c,t,s^N}\right) \tag{82}$$
$$+ \mathbb{E}_0 \sum_{t,s^N} \beta^t S_{t,s^N}\left(-\gamma g(a_{t,s^N}) + \lambda_{t,s^N}\gamma g'(a_{t,s^N})\right).$$

Solving the Ramsey program. The Ramsey program then consists in maximizing J_γ in (82) over $((a_{t,s^N}, c_{t,s^N}, l_{t,s^N})_{s^N}, \phi_t, \tau_t)_{t\geq 0}$ subject to the relevant constraints. As in the baseline case, this problem can thus be seen as depending only on saving choices (a_{t,s^N}) and the replacement rate ϕ_t. The FOC for ϕ_t is independent of penalty functions and is identical to (18). We only focus here on the FOC with respect to (a_{t,s^N}), which is after some algebraic manipulations:

$$\frac{\Psi_{t,s^N}}{\beta} = \sum_{\tilde{s}^N}\mathbb{E}_t\Bigg[\Psi_{t,\tilde{s}^N}\Bigg((1+r_{t+1})\Pi_{t+1,s^N\tilde{s}^N} + S_{t+1,\tilde{s}^N}(F_{KK,t+1} + \frac{\varphi L_{t+1}\frac{F^2_{KL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}})\tilde{a}_{t,s^N}$$
$$+ (1+\varphi)S_{t+1,\tilde{s}^N}\tilde{\tau}_{t+1} l_{\tilde{s}^N,e,t+1}\tilde{y}_0^N \frac{F_{KL,t+1}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}}\Bigg)\Bigg]$$
$$+ \mathbb{E}_t \sum_{\tilde{s}^N} S_{t+1,\tilde{s}^N}\left(F_{KK,t+1} + \frac{\varphi L_{t+1}\frac{F^2_{KL,t+1}}{F_{L,t+1}}}{1-\varphi L_{t+1}\frac{F_{LL,t+1}}{F_{L,t+1}}}\right)\tilde{\lambda}_{t+1,\tilde{s}^N}\xi_{\tilde{s}^N}U_{c,\tilde{s}^N,t+1}$$
$$- \gamma g'(a_{t,s^N}) + \lambda_{t,s^N}\gamma g''(a_{t,s^N}).$$

Due to the exterior penalty function, for histories for which $a_{t,s^N} \leq -\bar{a}$, $a_{t,s^N} \to -\bar{a}$ as $\gamma \to \infty$ (Assumption B has to be fulfilled at any optimal equilibrium). In addition, the previous equality implies that $\lambda_{t,s^N} \to 0$ when $\gamma \to \infty$ and that the constraints "disappear", as was shown in Section E.1. See Luenberger and Ye (2016) for a proof in a more general case. Finally, for histories for which $a_{t,s^N} > \bar{a}$, the previous constraint converges to the same constraint as in the initial truncated program (given in (32)). In our truncated equilibrium without penalty function, we have $\lambda_{t,s^N} = 0$ and $a_{t,s^N} = -\bar{a}$ for the credit-constrained history. As the FOC for ϕ_t is the same in our initial problem and in the problem with a penalty function, our allocation is therefore a solution of the limit of program with infinitely concave penalty functions. This concludes the proof.

E.3 Generalizing the truncation theory to non-GHH utility functions

We generalize the truncation method to a separable instantaneous utility function, $U(c,l) = u(c) - v(l)$, instead of the GHH utility function of the main text. The functions u and v are supposed to be continuous, twice differentiable, increasing, and concave in both arguments. For the sake of clarity, the presentation follows the same structure as in the main text.

E.3.1 The set-up

Besides this more general utility function, the rest of the economy is strictly similar to the economy presented in Section 2.

The agent's program can now be written as:

$$\max_{\{c_t^i, l_t^i, a_t^i\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (u(c_t^i) - v(l_t^i 1_{e_t^i=e} + \zeta_{y_t^i} 1_{e_t^i=u})) \tag{83}$$

$$c_t^i + a_t^i = (1+r_t)a_{t-1}^i + ((1-\tau_t)l_t^i 1_{e_t^i=e} + \phi_t \bar{l}_t \left(y_t^i\right) 1_{e_t^i=u})y_t^i w_t, \tag{84}$$

$$a_t^i \geq -\bar{a},\ c_t^i > 0,\ l_t^i > 0. \tag{85}$$

Compared to (5)–(7), the objective function reflects the different utility function, but the constraints are unchanged. The agents' first-order conditions become:

$$u'(c_t^i) = \beta \mathbb{E}_t \left[(1+r_{t+1})u'(c_{t+1}^i)\right] + \nu_t^i, \tag{86}$$

$$v'(l_t^i) = (1-\tau_t)w_t y_t^i u'(c_t^i), \tag{87}$$

where the Euler equation (87) obviously only holds for employed agents.

Market clearing conditions (12) are unchanged, and the only difference is that Euler equations for consumption and labor are more involved than in the GHH case. In particular, the individual labor supply is no longer a linear function of productivity.

E.3.2 The truncated model

Aggregation. As in the GHH economy, we aggregate individual allocations along the truncated history of agents (i.e., their individual idiosyncratic history over the last N periods, N being a given truncation length). The process is very similar (for the budget constraint, and market clearing conditions for instance) and the only difference concerns Euler equations. The aggregation of (88) and (89) yields:

$$\xi_{s^N,t} u'(c_{t,s^N}) = \beta \mathbb{E}_t \Big[\sum_{\tilde{s}^N \succeq s^N} \Pi_{t+1,s^N,\tilde{s}^N} \xi_{\tilde{s}^N,t} u'(c_{t+1,\tilde{s}^N})(1+r_{t+1}) \Big] + \nu_{t,s^N}, \tag{88}$$

$$\xi^l_{s^N,t} v'(l_{t,s^N}) = (1-\tau_t) w_t y_0^N \xi_{s^N,t} u'(c_{t,s^N}), \tag{89}$$

where $(\xi_{s^N,t})_{s^N}$ have the same interpretation as in the GHH economy and $(\xi^l_{s^N,t})_{s^N}$ are their counterparts that are due to the non-linearity of the Euler equation (89) for labor (unlike in the GHH economy).

Steady-state economy. Prices and allocations at the steady state of the aggregated economy are characterized by the Euler equations for consumption and labor (88) and (89), the collection of (unchanged) budget constraints (24), and the (unchanged) dynamics of history sizes (21). We have the following for the Euler equations:

$$\xi_{s^N} u'(c_{s^N}) = \nu_{s^N} + \beta(1+r) \sum_{\tilde{s}^N \succeq s^N} \Pi_{s^N,\tilde{s}^N} \xi_{\tilde{s}^N} u'(c_{\tilde{s}^N}), \tag{90}$$

$$\xi^l_{s^N} v'(l_{s^N}) = (1-\tau_t) w_t y_0^N \xi_{s^N} u'(c_{s^N}), \tag{91}$$

As in the GHH economy, the steady-state equilibrium is further characterized by some unchanged equations: market clearing equations (28), UI scheme budget balance (14), and factor prices (3).

Computing the ξs and ξ^ls. Determining the ξs and ξ^ls follows exactly the same logic as in the GHH economy, using the Bewley allocations, in particular of consumption and labor. The ξs (for consumption) are determined such that the aggregated consumption levels (for each history) verify the steady-state consumption Euler equation (90). The difference compared with the GHH case is that, due to the separability of the instantaneous utility function in consumption and labor, this operation only requires consumption allocations of the Bewley model. However, this separability also enables us to determine the ξ^ls. This computation is straightforward and for each history, the ξ_{s^N} for each history s^N is computed using the Euler equation for labor (91).

We also have a similar convergence result to that of Proposition 2.

Proposition 7 (Convergence of allocations) *With similar notation to Proposition 2, we have the following convergence result for allocations:*

$$(c_{t,s^N}, a_{t,s^N}, l_{t,s^N})_{s^N} \longrightarrow_N (c_t(s^\infty), a_t(s^\infty), l_t(s^\infty))_{s^\infty \in \mathcal{S}^\infty}, \text{ almost surely.}$$

Similarly, for preference shifters: $\xi_{s^N} \longrightarrow_N 1$ *and* $\xi^l_{s^N} \longrightarrow_N 1$, *almost surely.*

In other words, constructing a truncated model in the presence of a separable utility function is possible and its main properties do still hold.

The dynamics. As in Section 4.2, we make the assumption that in the presence of aggregate risk, the ξs and ξ^ls remain constant and equal to their steady-state values. Similarly, the set of credit-constrained histories is assumed to be time-invariant. The resulting truncated model in the presence of aggregate shocks is then characterized by Euler equations (88) and (89) with $\xi_{s^N,t} = \xi_{s^N}$ and $\xi^l_{s^N,t} = \xi^l_{s^N}$ (the ξs remain equal to their steady-state values), as well as by the budget constraint (24), the market clearing conditions (28), the factor prices (3), and the UI budget constraint (14) (which remain unchanged compared to the GHH economy).

E.3.3 The Ramsey program

Formulation. We now formulate the Ramsey program in the presence of the separable utility function. The Ramsey program of equation (15) becomes:

$$\max_{((a_{t,s^N},c_{t,s^N},l_{t,s^N})_{s\in\mathcal{S}^N},\phi_t,\tau_t,r_t,w_t)_{t\geq 0}} \mathbb{E}_0\Big[\sum_{t,s^N}\beta^t S_{t,s^N}(\xi_{s^N}u(c_{t,s^N})-\xi^l_{s^N}v(\hat{l}_{t,s^N}))\Big], \tag{92}$$

subject to Euler equations for (88) and (89), as well as the same set of equations as in the GHH case: (i) the budget constraints (24), (ii) the UI scheme budget balance (14), (iii) the market clearing constraints (28), and finally (iv) the factor prices (3).

Ramsey first-order conditions. To simplify the derivation of first-order conditions, we will also assume that the labor supply $(l_{t,s^N})_{s^N\in\mathcal{S}^N}$ is a choice variable, in addition to savings choices $(a_{t,s^N})_{s^N\in\mathcal{S}^N}$ and the replacement rate ϕ_t (already present in the GHH case).

The Lagrange multiplier for the consumption Euler equation is still $\beta^t S_{t,s^N}\lambda_{t,s^N}$, while the one for the labor Euler equation is $\beta^t S_{t,s^N}\mu_{t,s^N}$. We introduce $\tilde{\Psi}_{t,s^N}$, defined as:

$$\tilde{\Psi}_{t,s^N}=\Psi_{t,s^N}+(1-\tau_t)F_{L,t}\mu_{t,s^N}\xi_{s^N}u''(c_{t,s^N}),$$

which reflects the value of liquidity. In addition to the GHH case, it accounts for the fact that the labor supply diminishes if consumption increases. The FOC for saving choices is:

$$\begin{aligned}\tilde{\Psi}_{t,s^N}&=\beta\sum_{\tilde{s}^N}\mathbb{E}_t\left[\Big((1+r_{t+1})\Pi_{t+1,s^N,\tilde{s}^N}+F_{KK,t+1}S_{t+1,\tilde{s}^N}\tilde{a}_{\tilde{s}^N,t}\Big)\tilde{\Psi}_{t+1,\tilde{s}^N}\right] \\ &+\beta F_{KL,t+1}\sum_{\tilde{s}^N}\mathbb{E}_t\left[S_{t+1,\tilde{s}^N}\tilde{\Psi}_{t+1,\tilde{s}^N}((1-\tau_{t+1})1_{\tilde{e}_0^N=e}+\frac{S_{t+1,e}}{S_{t+1,u}}\tau_{t+1}1_{\tilde{e}_0^N=u})\tilde{y}_0^N l_{t+1,\tilde{s}^N}\right] \\ &+\beta\sum_{\tilde{s}^N}\mathbb{E}_t\Big[S_{t+1,\tilde{s}^N}\Big(F_{KK,t+1}\tilde{\lambda}_{t+1,\tilde{s}^N}+(1-\tau_{t+1})F_{KL,t+1}\mu_{t+1,\tilde{s}^N}\xi_{\tilde{s}^N}\Big)\xi_{\tilde{s}^N}u'_{t+1,\tilde{s}^N}\Big].\end{aligned} \tag{93}$$

The two first lines are very similar to the GHH case, while the last one includes a term that is specific to our separable utility function. The FOC with respect to labor supply is:

$$\begin{aligned}&\frac{\xi^l_{s^N}}{y_0}\Big(v'(l_{t,s^N})+\mu_{t,s^N}v''(l_{t,s^N})\Big)-\left((1-\tau_t)1_{\tilde{e}_0=e}+\frac{S_{t,e}}{S_{t,u}}\tau_t 1_{\tilde{e}_0=u}\right)F_{L,t}\xi_{s^N}\tilde{\Psi}_{t,s^N} \\ &=\sum_{\tilde{s}^N\in\mathcal{S}^N}S_{t,\tilde{s}^N}\xi_{\tilde{s}^N}\tilde{\Psi}_{t,\tilde{s}^N}\left(F_{KL,t}\tilde{a}_{s^N,t-1}+((1-\tau_t)1_{\tilde{e}_0=e}+\frac{S_{t,e}}{S_{t,u}}\tau_t 1_{\tilde{e}_0=u}y_0)F_{LL,t}l_{t,\tilde{s}^N}\right) \\ &+\sum_{\tilde{s}^N}S_{t,\tilde{s}^N}\Big(F_{KL,t}\tilde{\lambda}_{t,\tilde{s}^N}+(1-\tau_t)F_{LL,t}\mu_{t,\tilde{s}^N}\Big)\xi_{\tilde{s}^N}u'_{t,\tilde{s}^N}.\end{aligned} \tag{94}$$

Finally, the first-order equation with respect to the replacement rate is:

$$\sum_{s^N\in\mathcal{S}^N}S_{t,s^N}\mu_{t,s^N}\xi_{s^N}u'(c_{t,s^N})=\sum_{s^N\in\mathcal{S}^N}S_{t,s^N}\xi_{s^N}\tilde{\Psi}_{t,s^N}(-1_{e_0^N=e}+\frac{S_{t,e}}{S_{t,u}}1_{e_0^N=u})y_0^N l_{t,\tilde{s}^N}, \tag{95}$$

which balances the benefits of a higher replacement rate with its cost. The proofs for expressions (93)–(95) follows the exact sames lines as the one of Section 4.4 (with GHH utility function) and is skipped for the sake of conciseness.

The algorithm for simulation the Ramsey solution remains the same as the one of Section 4.6 in the GHH case, except that both the ξs and ξ^ls have to be computed using the steady-state allocations (Step 4.a).

E.3.4 Conclusion

Section E.3 has shown that our truncation method can be readily extended to a more general utility function. The only difference is that due to the non-linearity of the Euler equation for labor supply, we need to include ξs that are specific to the labor Euler equation. The rest (including the Ramsey problem) is mostly unchanged.

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